Intermediate 1 **Jan 7, 2024**

Euclidean Geometry

Geometry is, along with arithmetic, one of the oldest branches of mathematics. The geometry that we are most familiar with is called Euclidean geometry, named after the famous Ancient Greek mathematician Eucid.

Euclid

We know very little about this man renowned as the "Father of Geometry". His book, called "Elements", was the first geometry textbook on this planet. The most successful scientific publication of all time, it was in active use from the moment of its creation until the end of the 19th century, about 2,200 years altogether! Many of the results presented in the "Elements" were discovered by mathematicians preceding Euclid. It was his achievement to organize them in a logically coherent manuscript, including a system of rigorous proofs that remains the basis of mathematics 23 centuries later.

1 Points and Lines

Here is how Euclid defines points and lines:

Definition 1. *A **point** is that which has no part.*

Definition 2. *A **line** is a length without breadth. A **straight line** is a line which lies evenly with points on itself.*

Definition 3. *A **circle** is a plane figure contained by a single line, called the circumference, such that all straight line segments from one point, called the center, to the circumference are equal to each other.*

Problem 1.1. *Using similar words, define what is a surface and a plane.*

Besides the above definitions, Euclid also proposed a few assumptions, known as **postulates**. These can be thought of as the basic rules of Euclidean geometry.

Postulate 1: *A straight line may be drawn from any one point to any other point.*

Postulate 2: *A line segment can be produced indefinitely in a straight line.*

In other words, we can draw a line segment with arbitrary length within a straight line.

Problem 1.2. *M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{BM}$. If $BN = 4$, then what is AB?*

Problem 1.3. *Points A, B, C, D, and E are five points on a line segment with endpoints A and E. The points are in the order listed above from left to right such that $CD = AB/2$, $BC = CD/2$, $AB = AE/2$, and $AE = 12$. What is the length of $\overline{AD}$?*

Postulate 3: *A circle can be drawn with any given center and radius.*

Problem 1.4. *Two circles and three straight lines lie in the same plane. If neither the circles nor the lines are coincident (meaning the two circles are different and the three lines are all different lines), what is the maximum possible number of points at which at least two of the five figures intersect?*

Definition 4. *A pair of* ***parallel lines*** *are two straight lines which, being in the same plane, and after being extended to infinity in each direction, meet with one another in neither of these directions.*

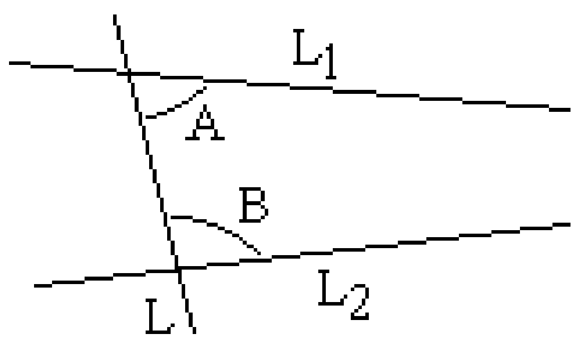
Definition 5. *When a straight-line stood upon another straight line, making the adjacent angles equal to one another, each of the equal angle is called a* ***right angle****, and the two lines are called* ***perpendicular*** *to each other.*

Problem 1.5. *Draw two lines that are parallel to each other. Then draw two lines that are perpendicular to each other.*

Postulate 4: *All right angles are equal.*

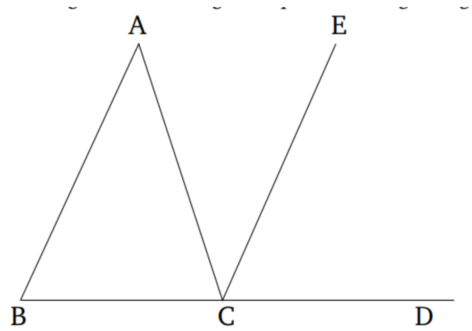
Postulate 5: *If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.*

In other words, if the angle A and B in the image below adds up to less than $180°$, then the line L_1 and L_2 will intersect with each other after extending to infinity.



Problem 1.6. *Use the notation $L_1||L_2$ to denote L_1 parallel to L_2. In the image above, suppose $L_1||L_2$. Prove that $\angle A + \angle B = 180°$.*

Problem 1.7. *In the figure below, suppose $AB || CE$.*



Use the postulates above to prove that

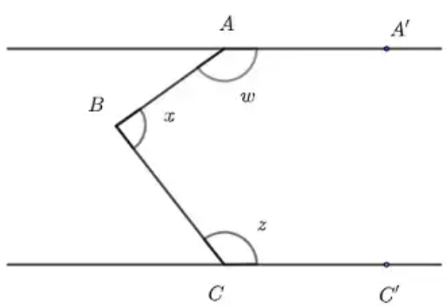
- $\angle BAC = \angle ACE$
- $\angle ABC = \angle ECD$

Problem 1.8. *Use the previous problem, prove that the sum of interior angles in any triangle is* $180°$.

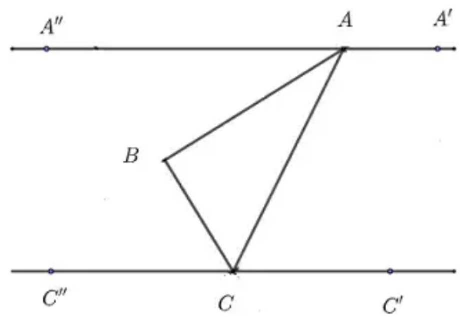
Problem 1.9. *The angles in a triangle are in the ratio* $1 : 2 : 3$. *What are the measures of the angles?*

Problem 1.10. *Prove the converse of Postulate 5: suppose a straight line falling on two straight lines makes the interior angles on the same side sum up to* $180°$*, then the two straight lines are parallel.*

Problem 1.11. *In the figure below,* AA' *is parallel to* CC'*. The size* w *of* $\angle A'AB$ *is equal to* $135°$ *and the size* z *of* $\angle C'CB$ *is equal to* $147°$*. Find* x*.*



Problem 1.12. *In the figure below lines $A'A''$ and $C'C''$ are parallel. AB is the bisector of angle CAA'' and BC is the bisector of angle ACC''. Show that the $\angle ABC = 90°$.*



Problem 1.13. *Lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are parallel, and $\overleftrightarrow{TV} \perp \overleftrightarrow{PQ}$. If $\overleftrightarrow{TV}$ intersects $\overleftrightarrow{PQ}$ at X and $\overleftrightarrow{RS}$ at Y, find $\angle RYX$.*

2 Similar Triangles

Definition 6. *Triangle ABC is said to be similar to triangle $A_1B_1C_1$ (we write $\triangle ABC \sim \triangle A_1B_1C_1$) if and only if one of the following equivalent conditions is satisfied:*

a) $AB : BC : CA = A_1B_1 : B_1C_1 : C_1A_1$

b) $AB : BC = A_1B_1 : B_1C_1$ *and* $\angle ABC = \angle A_1B_1C_1$

c) $\angle ABC = \angle A_1B_1C_1$ *and* $\angle BAC = \angle B_1A_1C_1$

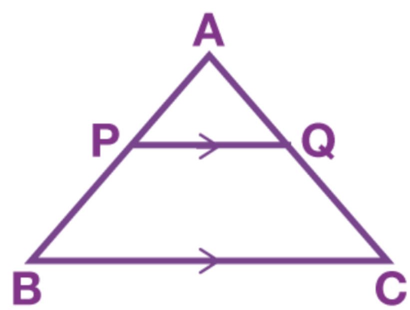
Problem 2.1. *Draw two triangles that are similar but with different sizes. Explain why they satisfy each of the conditions above.*

Problem 2.2. *Given that $\triangle ABC \sim \triangle YXZ$, which of the statements below must be true?*

1. $AB/YX = AC/YZ$
2. $AB/BC = YX/XZ$
3. $AB/XZ = BC/YX$
4. $(AC)(YX) = (YZ)(BA)$
5. $BC/BA = XY/ZY$

Problem 2.3. *$\triangle ABC \sim \triangle ADB$, $AC = 4$, and $AD = 9$. What is AB?*

Problem 2.4. *In the figure below, suppose P is the midpoint of the line segment AB, and Q is the midpoint of AC. Prove that $PQ||BC$*



Problem 2.5. *Consider heights AA_1 and BB_1 in acute triangle ABC. Prove that $A_1C \cdot BC = B_1C \cdot AC$.*

Problem 2.6. *X and Y are on sides $\overline{PQ}$ and $\overline{PR}$, respectively, of $\triangle PQR$ such that $\overline{XY} \mid\mid \overline{QR}$. Given $XY = 5$, $QR = 15$, and $YR = 8$, find PY.*

Problem 2.7. *Prove that the midpoints of the sides of an arbitrary quadrilateral are vertices of a parallelogram. For what quadrilaterals this parallelogram is a rectangle, a rhombus, a square?*

Problem 2.8. *Point K lies on diagonal BD of parallelogram $ABCD$. Straight line AK intersects lines BC and CD at points L and M, respectively. Prove that $AK^2 = LK \cdot KM$.*

Problem 2.9. *Through an arbitrary point P on side AC of $\triangle ABC$ straight lines are drawn parallelly to the triangle's medians AK and CL. The lines intersect BC and AB at E and F, respectively. Prove that AK and CL divide EF into three equal parts.*