Intermediate 1 **Dec 10, 2023**

Competition!

1 10 Points Questions

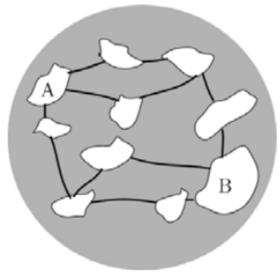
Problem 1.1. *What is the probability of getting exactly 2 heads when tossing 6 fair coins?*

Problem 1.2. *In how many ways you can choose 3 people out of 24 people, assuming order does not matter and no replacement?*

Problem 1.3. *On the island of knights and knaves, knights always tell the truth and knaves always lie. You ran into two individuals on the island, Alice and Bob. You asked Alice: "Are you a Knave, yes or no?" The Alice mumbled something incomprehensible. So, you asked Bob what she had said. He answered, "She said no." Figure out who Alice and Bob are.*

Problem 1.4. *Ana starts lighting a candle every 10 minutes. Each candle lasts 40 minutes. After 55 minutes, how many candles will be lit?*

Problem 1.5. *How many integers from 1 to 1000 are not divisible by 9?*



Problem 1.6. *The 10 islands are connected by 12 bridges:*

All bridges are open for traffic. What is the minimum number of bridges that need to be closed off, so that the traffic between A and B comes to a halt?

Problem 1.7. *Write the image of the following function*

$$f : \{1, 2, 3, 4\} \to \mathbb{R}$$

given by $f(x) = 2x + 4$.

Problem 1.8. *How many elements are in the following set?*

$$\{1, 2, 3, 4\} \times \{5, 6, 7, 8, 9\}$$

Problem 1.9. *In a bag there are only red and green marbles. If one randomly takes out five marbles, there is at least one red one. If one randomly takes out six marbles, there is at least one green one. What is the maximum number of marbles in the bag?*

Problem 1.10. *Let the set* $A = \{\text{Red}, \text{Blue}, \text{Green}\}$ *and* $B = \{\text{White}, \text{Black}\}$. *What is* $A \cup B$ *and* $A \cap B$*?*

Problem 1.11. *What is the cardinality of* $\mathcal{P}(A)$, *where* A *is the set* $\{2, 4, 6, 8\}$*?*

2 20 Points Questions

Problem 2.1. *Big Al the ape ate 100 delicious yellow bananas in total from May 1 through May 5. Each day he ate six more bananas than on the previous day. How many delicious bananas did Big Al eat on May 5?*

Problem 2.2. *How many different isosceles triangles have integer side lengths and perimeter 23?*

Problem 2.3. *Alice has a special coin. Each times she tosses the coin, there is a $\frac{4}{5}$ chance that she gets a head and $\frac{1}{5}$ chance that she gets a tail. What is the probability of getting exactly 3 heads when she toss the coin 5 times?*

Problem 2.4. *Igor writes down all results of the quarter finals, the semi finals and the final of a tennis tournament. The results are listed in random order:*

Bert beats Anton, *Carl beats Damien,* *Glen beats Henry,*
Glen beats Carl, *Carl beats Bert,* *Edon beats Fred,*
Glen beats Edon.

Who is playing in the final?

Problem 2.5. *How many integers from 1 to 100 are not divisible by 7 or 11?*

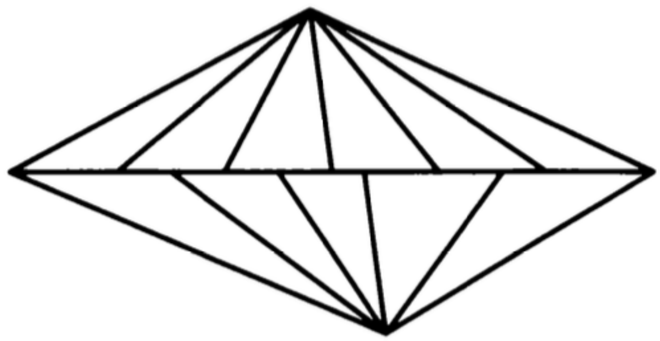
Problem 2.6. *Five friends bake gingerbread cake and subsequently meet up for a tasting session. Each one slices their gingerbread and gives one slice to each of their friends. Each person then eats all of the slices they were given, after which the total number of gingerbread slices halves. How many gingerbread slices did the five friends have to start with?*

Problem 2.7. *Five boxes contain 2, 3, 4, 7 and 15 balls respectively. Peter wants to redistribute the balls into the boxes so that any box has twice or half the number of balls in one of the remaining boxes. What is the minimum number of balls he can move to achieve his goal?*

Problem 2.8. *A bowling tournament involves 8 participants, who are ranked #1 through #8 before the tournament begins. The #8 bowler plays a game head-to-head against the #7 player. The loser of this game comes in 8th place in the tournament. The winner bowls a game against the #6 player. The loser of this game is designated as 7th place in the tournament, and the winner plays head-to-head against player #5, and so on. Eventually someone plays #1 for the championship. Given a fixed starting lineup of 8 bowlers, how many different outcomes of the tournament are possible?*

Problem 2.9. *Let* $x = 0.1234567891011121314\ldots997998999$, *where* x *is formed by writing all of the integers from 1 to 999 after the decimal point. What is the 2005th digit to the right of the decimal point?*

Problem 2.10. *How many triangles are in the following figure:*



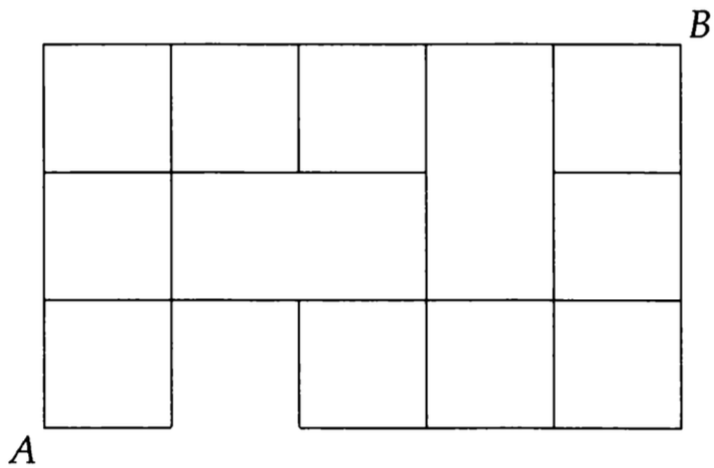
Problem 2.11. *Nine parallel lines in a plane intersect a set of n parallel lines that go in another direction. The lines form a total of 360 parallelograms, many of which overlap each other. What is n?*

Problem 2.12. *Janie is a house painter. She has been given a contract to paint any 3 of the houses that she wants on a street that contains 5 houses. Janie has 4 different colors of paint, in unlimited supply. She can use only one color on any given house, but for each house, she gets to pick which color to use. In how many different ways can Janie fulfill her contract?*

Problem 2.13. *1000 unit cubes ($1 \times 1 \times 1$ cubes) are glued together to form a $10 \times 10 \times 10$ cube. At most how many of these unit cubes are visible from a single point in space?*

Problem 2.14. *I have tiled my square bathroom wall with congruent square tiles. All the tiles are red, except those along the two diagonals, which are all blue (i.e. the corners are blue and all the tiles along the diagonals between each pair of opposite corners are blue). If I used 121 blue tiles, how many red ones did I use?*

Problem 2.15. *In the grid shown below, how many paths are there from A to B using only steps up or to the right? Note that a path cannot traverse over a missing edge!*

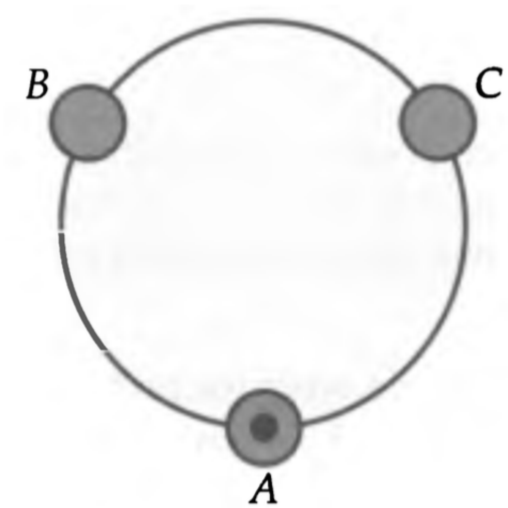


Problem 2.16. *What is the probability that a random rearrangement of the letters in the word 'MATHEMATICS' will begin with the letters 'MATH'?*

Problem 2.17. *Points A, B, and C are arranged clockwise in that order on a circle, as shown below. We place a marker on point A. We roll a 6-sided die and move the marker as follows:*

- *If the die shows a 1 or a 2 , stay put.*
- *If the die shows a 3 or a 4, move one step clockwise (for example, from A to B).*
- *If the die shows a 5 or a 6, move one step counterclockwise (for example, from A to C).*

What is the probability that after 8 moves the marker is at point A?



3 50 Points Questions

Problem 3.1. *Alex has a 1 m long and a 2 m long rope. He cuts up both ropes so that all pieces are of equal length. What is the set* $A \subseteq \mathbb{N}$ *of all possible numbers of pieces he can get after the cuts? (Hint: A is very large)*

Problem 3.2. *Each face of a cube is given a single narrow stripe painted from the center of one edge to the center of the opposite edge. The choice of the edge pairing is made at random and independently for each face. What is the probability that there is a continuous stripe encircling the cube?*

Problem 3.3. *Write a bijection between the set of natural numbers divisiable by 3 and the set* $\mathbb{Z}$.

Problem 3.4. *Spiffy the Spider has 8 legs, each of which is a different color. Spiffy has a sock and a shoe for each of his 8 legs (so that he has one sock and one shoe in each of the 8 colors). In how many ways can Spiffy put on his socks and shoes, if each sock and shoe must go on the leg with the matching color, and the sock must go on before the shoe on each leg?*

Problem 3.5. *In terms of* n*, how many solutions in non-negative integers are there to* $x + y + z = n$*?*

4 100 Points Questions

Problem 4.1. *Each of 2010 boxes in a line contains a single red marble, and for* $1 \leq k \leq 2010$*, the box in the* k*th position also contains* k *white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let* $P(n)$ *be the probability that Isabella stops after drawing exactly* n *marbles. What is the smallest value of* n *for which* $P(n) < \frac{1}{2010}$*?*

Problem 4.2. *Three friends Xavier, Yul, and Zeb are having a 3-way paintball duel. Xavier is a poor shot, hitting his target with probability 1/3. Yul is a pretty good shot, hitting his target with probability 2/3. Zeb is a marksman, always hitting his target. The rules of the duel are as follows: in turn, each person will get one shot at a target of his choice. Once you're hit, you're out of the game. The last person remaining wins. Since he's the worst shot, Xavier gets to go first, followed by Yul, followed by Zeb. What is Xavier's best strategy, and what is his probability of winning?*