Pigeons, Holes, and a Principle

Math Circle (Intermediate)

October 21, 2012

SPECIAL DIRECTIONS: For any problem that asks you to "explain" something, you MUST write a complete sentence.

1. A bag contains 5 black beads and 5 white beads. How many beads do you need to draw (without looking) to ensure that you have two beads of the same color?

2. One million pine trees grow in a forest. It is known that no pine tree has more than 600,000 pine needles on it. **Explain** why there are two pine trees in the forest that have the same number of pine needles.

3. Twenty-five crates of apples are delivered to a store. Each crate contains one of three types of apples. **Explain** why among the 25 crates there must be at least 9 containing the same type of apples.

Whether or not you were aware of it when you solved Problems #1-3, you were directly using the **pigeonhole principle**. The basic pigeonhole principle states that if n items are put into m pigeonholes with $n > m$, then at least one pigeonhole must contain more than one item.

When we solve problems using this principle, we usually need to first determine what are the "pigeons" and what are the "holes". If the number of "pigeons" is larger than the number of "holes", at least two "pigeons" will be in the same "hole".

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Figure 1: Nine pigeons and nine holes.

In Figure 1 above, for example, the holes are the boxes and the pigeons are...well, pigeons. We can easily count that there are nine holes and nine pigeons. However, there are no empty holes remaining, so if we introduce just one more pigeon, it must share a hole with one of the existing pigeons.

From this point on, for any problem in which you use the pigeonhole principle, write down what are the pigeons and what are the holes, and how many of each there are.

4. Let's take a quick look back at the problems we just solved.

   (a) What were the holes and what were the pigeons in Problem 1? How many holes and how many pigeons are there?

	Item	Quantity
Holes		
Pigeons		

   (b) In Problem 2?

	Item	Quantity
Holes		
Pigeons		

   (c) In Problem 3?

	Item	Quantity
Holes		
Pigeons		

5. Prove the pigeonhole principle by contradiction.

6. 100 letters arrived for 20 people. Show that someone got more than 4 letters. First, fill in the table below:

	Item	Quantity
Holes		
Pigeons		

   Now draw a grid analogous to the one in the picture (Figure 1) to represent the holes, and use tally marks to represent the pigeons. Then use the pigeonhole principle to finish your proof.

7. There are 25 students in the Math Circle Intermediate group. Show that there are three students from this group that have their birthday in the same month. Use the pigeonhole principle and fill in the following table:

	Item	Quantity
Holes		
Pigeons		

8. Given any 3 positive integers, show that you can always select two of them so that their sum is even. Use the pigeonhole principle and fill in the following table:

	Item	Quantity
Holes		
Pigeons		

9. UCLA's beautiful Royce Hall auditorium seats approximately 1,300 people. For a sold out event, is it true that there will always be a pair of people present who have the same first and last initials? Use the pigeonhole principle to prove it!

	Item	Quantity
Holes		
Pigeons		

10. After Math Circle last Sunday, Ralph walked back to his car and found a parking ticket on his windshield. The policeman had run a search on Ralph's vehicle and wrote down the last four digits of Ralph's social security number and Ralph's first and last initials on his ticket (but nothing else).
Ralph challenged the ticket in state traffic court, telling the judge, "That wasn't me who parked the car illegally!"
The judge laughed. "Of course it was you, Ralph. Under penalty of perjury, what are the last four digits of your social security number?" Ralph reluctantly provided the four numbers, which matched exactly.
"Aha!" Ralph said. "But there are 40 million people living in California. You can't prove that it was me!"
The judge, who happened to be a mathematician, replied, "You're absolutely right," and tore up Ralph's ticket. **Why**?

	Item	Quantity
Holes		
Pigeons		

11. For the following four questions, assume that each problem in the math olympiad was solved by exactly one student.

    (a) Four students solved a total of 21 problems in a math olympiad. Use the pigeonhole principle to explain why at least one student must have solved at least five problems.

	Item	Quantity
Holes		
Pigeons		

(b) Now suppose five students solved a total of 21 problems in a math olympiad. There is at least one student who solved exactly one problem, and at least one student who solved exactly two problems. Show that at least one student solved at least five problems.

	Item	Quantity
Holes		
Pigeons		

(c) Now suppose ten students solved a total of 50 problems in a math olympiad. There is at least one student who solved exactly one problem, at least one student who solved exactly two problems, and at least one student who solved exactly three problems. Prove that there is at least one student who solved at least five problems.

	Item	Quantity
Holes		
Pigeons		

(d) What is the minimum number of problems that could have been on the math olympiad in part (c) if all of the same conditions must be satisfied?

12. There are n students in the LA Math Circle. On the first day of class, some of the students shook hands with other students.

    (a) I then select a student at random. How many possibilities are there for the number of people whose hands that student shook?
    (Hint: Be careful – what happens if one person shakes *everyone* else's hand?)

    (b) Use the pigeonhole principle to show that there will always be two students who have shaken hands with the same number of people.

        i. What are the pigeons and holes, and how many of each are there?

	Item	Quantity
Holes		
Pigeons		

        ii. Complete the proof on your own!

13. Angela draws five points on the surface of a spherical orange in permanent marker. Is there *always* a way to cut this orange in half so that four of the points will lie on the same hemisphere? (Suppose a point exactly on the cut belongs to both hemispheres). Make sure to explicitly reference the pigeonhole principle in your answer.

14. Given that the value of the fraction (which does not contain a typo)

$$\frac{B \cdot L \cdot U \cdot E \cdot B \cdot E \cdot R \cdot R \cdot Y}{I \cdot C \cdot E \cdot C \cdot R \cdot E \cdot A \cdot M}$$

is a real number, find the value of the fraction. Every letter represents a different digit. Justify your answer!

15. Jeff picks four integers between 1 to 8.

    (a) Is it true that two of them will always add up to 9? If so, prove it! If not, provide a counterexample.

(b) What is the smallest number of integers between 1 and 8 that Jeff must pick so that two of them will add up to 9?

16. In a room there are 10 people, none of whom are older than 60 (ages are considered as whole numbers only) but each of whom is at least 1 year old.

    (a) How many groups can you form out of these 10 people?
    (A group must include at least one person. For now, suppose it's okay for groups to share members, as long as the groups aren't identical.)

    (b) What are the possible values of the sum of ages for an unspecified group of people?

    (c) Prove that one can always find two groups of people, *with no common members*, so that the sums of ages of all the people in each of the group are the same.

17. Use the pigeonhole principle to prove that the decimal expansion of a rational number $\frac{m}{n}$ is eventually repeating. (Note that if a decimal "ends", like 0.75, it can be considered repeating: .750000000...).

18. Suppose you are asked to convert a rational number $\frac{m}{n}$ to a decimal. What is the largest number of decimal places you would possibly have to write of the decimal representation before you could determine the period at which it repeats? **Prove it**! (Use the pigeonhole principle.)[1]

[1]Some problems are taken from:
D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"
D. Patrick "Introduction to Counting and Probability"