# NUMBER THEORY

**UCLA Math Circle**
12/10/23

1. (2022 AMC 10A #7) The least common multiple of a positive integer $n$ and 18 is 180, and the greatest common divisor of $n$ and 45 is 15. What is the sum of the digits of $n$?

2. (2022 AMC 10A #19) Define $L_n$ as the least common multiple of all the integers from 1 to $n$ inclusive. There is a unique integer $h$ such that
$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{17}=\frac{h}{L_{17}}$$
What is the remainder when $h$ is divided by 17?

3. (2021 AMC 10B Problem 5) The ages of Jonie's four cousins are distinct single-digit positive integers. Two of the cousins' ages multiplied together gives 24, while the other two multiply to 30. What is the sum of the ages of Jonie's four cousins?

4. (2021 AMC 12B Problem 7) Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of $N$ to the sum of the even divisors of $N$ ?

5. (2021 AMC 10B Problem 13) Let $n$ be a positive integer and $d$ be a digit such that the value of the numeral $\underline{32d}$ in base $n$ equals 263 , and the value of the numeral $\underline{324}$ in base $n$ equals the value of the numeral $\underline{11d1}$ in base six. What is $n + d$?

6. (2021 AMC 10B Problem 16) Call a positive integer an uphill integer if every digit is strictly greater than the previous digit. For example, 1357, 89, and 5 are all uphill integers, but 32, 1240, and 466 are not. How many uphill integers are divisible by 15?

7. (2021 AMC 12A Problem 3) The sum of two natural numbers is 17402. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

8. (2021 AMC 12A Problem 5) When a student multiplied the number 66 by the repeating decimal,

$$\underline{1}.\underline{abab}\ldots = \underline{1} \cdot \underline{\bar{a}b},$$

where $a$ and $b$ are digits, he did not notice the notation and just multiplied 66 times $\underline{1}.\underline{ab}$. Later he found that his answer was 0.5 less than the correct answer. What is the 2-digit number $\underline{ab}$?

9. (2021 AMC 10A Problem 11) For which of the following integers b is the base-b number $2021_b - 221_b$ not divisible by 3?

(A) 3 (B) 4 (C) 6 (D) 7 (E) 8

10. (2021 AMC 12A Problem 12) All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of $B$?

11. (2021 AMC 12A Problem 18) Let $f$ be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers $a$ and $b$. Suppose that $f$ also has the property that $f(p) = p$ for every prime number $p$. For which of the following numbers $x$ is $f(x) < 0$?

    (A) $\frac{17}{32}$ (B) $\frac{11}{16}$ (C) $\frac{7}{9}$ (D) $\frac{7}{6}$ (E) $\frac{25}{11}$

12. (2020 AMC 10B Problem 6) Driving along a highway, Megan noticed that her odometer showed 15951 (miles). This number is a palindrome, meaning it reads the same forward and backward. Then 2 hours later, the odometer displayed the next higher palindrome. What was her average speed, in miles per hour, during this 2-hour period?

13. (2020 AMC 10B Problem 7) How many positive even multiples of 3 less than 2020 are perfect squares?

14. (2020 AMC 10B Problem 12) The decimal representation of

    $$\frac{1}{20^{20}}$$

    consists of a string of zeros after the decimal point, followed by a 9 and then several more digits. How many zeros are in that initial string of zeros after the decimal point?

15. (2020 AMC 10B Problem 15) Steve wrote the digits $1, 2, 3, 4$, and $5$ in order repeatedly from left to right, forming a list of 10,000 digits, beginning $123451234512\ldots$ He then erased every third digit from his list (that is, the $3\mathrm{rd}, 6\mathrm{th}, 9\mathrm{th}, \ldots$ digits from the left), then erased every fourth digit from the resulting list (that is, the $4\mathrm{th}, 8\mathrm{th}, 12\mathrm{th}, \ldots$ digits from the left in what remained), and then erased every fifth digit from what remained at that point. What is the sum of the three digits that were then in the positions $2019, 2020, 2021$?

16. (2020 AMC 10B Problem 22) What is the remainder when $2^{202} + 202$ is divided by $2^{101} + 2^{51} + 1$?

17. (2020 AMC 10B Problem 24) How many positive integers $n$ satisfy

$$\frac{n + 1000}{70} = \lfloor \sqrt{n} \rfloor?$$

(Recall that $\lfloor x \rfloor$ is the floor function, the greatest integer not exceeding $x$.)

18. (2020 AMC 10B Problem 25) Let $D(n)$ denote the number of ways of writing the positive integer $n$ as a product

$$n = f_1 \cdot f_2 \cdots f_k$$

where $k \geq 1$, the $f_i$ are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as $6, 2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

19. (2020 AMC 10A Problem 24) Let $n$ be the least positive integer greater than 1000 for which

$$\gcd(63, n + 120) = 21 \quad \text{and} \quad \gcd(n + 63, 120) = 60.$$

What is the sum of the digits of $n$?