FALL 2023

OLGA RADKO MATH CIRCLE
ADVANCED 1
DECEMBER 10TH, 2023

You MAY use your old packets. If you no longer have them, we don't have copies, but you can ask us to provide any definition or result that appears in the packet and we'll tell you.

The rules are as follows:

- Each problem starts off being worth 10 points.
- Every time a team submits an incorrect solution to that problem, the team loses a point and the problem gains a point.
- At the end, the number of points each problem has will be evenly distributed across all teams that solved it.

1. Random Variables

Problem 1.1. (10 points) Suppose you are betting on the outcome of rolling a pair of fair six-sided dice. For a \$1 buy-in, you can either bet "low", "high", or "seven" and the following occurs:

- If you correctly guess that the number rolled is less than 7, you will win back \$2.
- If you correctly guess that the number rolled is greater than 7, you win back \$2.
- If you correctly guess that the number rolled is exactly 7, you win back \$4.

If you play this game many, many times, what is your expected profit per game if you always bet on "low"?

Solution. You are expected to lose a sixth of a dollar per game: $\mathbb{E}[X] = -\frac{1}{6}$.

Problem 1.2. (10 points) Suppose Experiment A gives a number with expected value 5 and variance 2. What is the expected value and variance if we:

- run Experiment A once, and multiply the result by 4?
- run Experiment A twice, and multiply the sum of the results by 2?
- run Experiment A four times, and add the results?

Solution.

- 20 and 32
- 20 and 16
- 20 and 8

actually (?)

- 20 and 32
- 20 and 4
- 20 and 1/2

Problem 1.3. (10 points) In the following game, there is a pot that begins with 0 coins. You flip a coin, and if it lands heads, then 100 coins are added to the pot. If you flip tails, the pot is emptied and you are assigned a "strike". You may stop the game at any point to walk away with all the coins in the pot, but if you get three strikes, the game ends and you walk away with nothing. What is the highest expected number of coins you can get from this game?

Solution. 115.625 coins

2. Metrics

Problem 2.1. (10 points) Find the Euclidean and taxicab distances between the following pairs of points in the plane A and B.

- $A = (2, 0)$ and $B = (-1, 4)$
- $A = (6, 8)$ and $B = (1, -4)$
- $A = (179, 240)$ and $B = (203, 247)$

Solution.
- 5 and 7
- 13 and 17
- 25 and 31

Problem 2.2. (10 points) Sketch the taxicab line segment between $(0, 4)$ and $(3, 0)$ in the plane.

Solution.

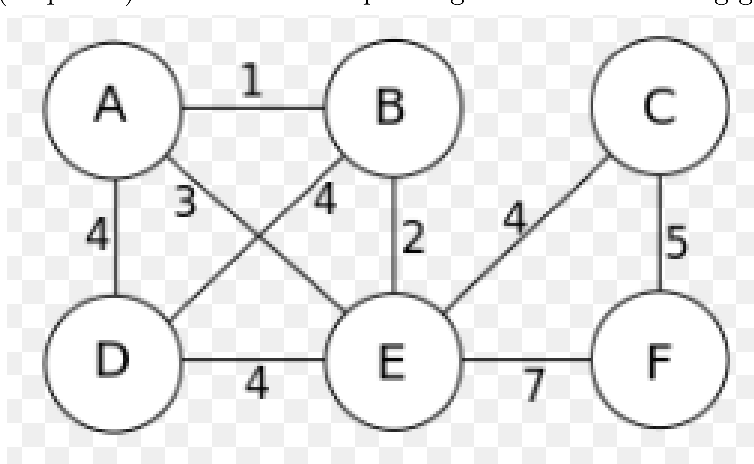
3. Combinatorial Optimization

Problem 3.1. (10 points) The following diagram is a map of the continental United States. We can form a graph of the states by saying two states are adjacent if they share a border of positive length (but not a corner). Given a Hamiltonian path of continental US states, which states can it *not* start at?

Solution. New York (because it splits the graph in two) and South Carolina (because it has degree 2) (?)

Answer should actually be all the states between NY and NH inclusive. Also, GA since it's surrounded by FL and SC which each have degree 2.

Problem 3.2. (10 points) Find a minimal spanning tree of the following graph.



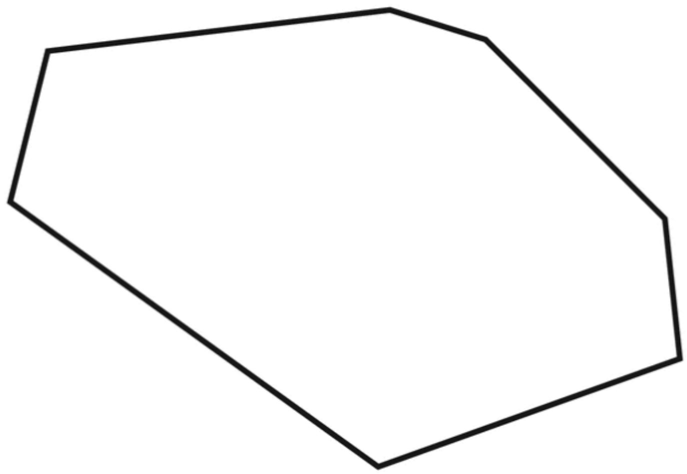
Solution.

4. Isoperimetric Inequality

Problem 4.1. (10 points) Find the probability that two random chords of a square in the plane intersect each other.

Solution. $\pi/8$

Problem 4.2. (10 points) Design an experiment to find the area of the following irregular heptagon using only a ruler to measure the lengths of the edges and to draw line segments.



Solution.

5. Information Theory

Problem 5.1. (10 points) We have 13 coins, one which we know is counterfeit. We don't know if the counterfeit coin is heavier or lighter than all of the real coins, which all weigh the same. Alice will let us use her balance to test the coins, but she will charge us \$10 for each use. Find the cheapest guaranteed way to identify the counterfeit coin.

Solution. This can be reduced to Problem 5 from last week by setting aside one coin and testing the other 12 using only three weighings.

Problem 5.2. (10 points) We are told that a fair coin has landed heads 20 times in a row. How many bits of information have we been told?

Solution. 20

6. Miscellaneous

Problem 6.1. (10 points) In base 10, the number 2023 ends in the digit 3. In how many bases b does 2023 end in the digit 3 when written in base b?

Solution. 2023 ends in 3 in base b if $b > 3$ and $2023 \cong 3 \pmod{b}$. Therefore b must divide $2020 = 2^2 \times 5 \times 101$, which has $3 \times 2 \times 2 = 12$ factors, 10 of which are greater than 3.

Problem 6.2. (10 points) Let S be the set of four-digit positive integers for which the sum of the squares of their digits is 17. Find the median of S.

Solution. 2302

Problem 6.3. (10 points) Find the smallest integer $n > 2023$ such that the quadratic polynomial $x^2 - 23x - n$ has integer root(s).

Solution. 2030

Problem 6.4. (10 points) Let R be a rhombus whose diagonals have length 1 and 2023. Let R' be the rotation of R by 90 degrees about its center. Find the ratio of the area of $R \cap R'$ to the area of $R \cup R'$.

Solution. 1/4045 or 1/2023?

Problem 6.5. (10 points) Let $Q(z)$ and $R(z)$ be the unique polynomials such that

$$z^{2023} + 1 = (z^2 + z + 1)Q(z) + R(z)$$

where the degree of $R(z)$ is less than 2. Find $R(z)$.

Solution. $z + 1$

Problem 6.6. (10 points) A group of 12 pirates agrees to divide a treasure chest of gold coins among themselves as follows. The k^{th} pirate to take a share takes $k/12$ of the coins that remain in the chest. Miraculously, the chest contained the smallest number of coins that would allow every pirate to receive a whole number of coins under this arrangement. How many coins did the last pirate receive?

Solution. 1925

Problem 6.7. (10 points) The product $(7)(777\ldots7)$ has digits that add up to 2021. How many 7's are in the second number?

Solution. 503

Problem 6.8. (10 points) Barbara has a business meeting at Corporation O, located at the point $(0, 0)$ in the coordinate plane. Since this is her first time visiting Corporation O, she has no idea how to get back home afterwards - all she knows is that her home is to the east, and not too far north or south. Suppose she lives at $(5, 3)$. Barbara will take one-unit steps either straight up, down, or right, while staying between $y = 0$ and $y = 3$ at all times, and not visiting any point more than once. How many different paths can she take to get home?

Solution. $4^5 = 1024$

Problem 6.9. (10 points) Regular polygons with 5, 6, 7, and 8 sides are inscribed in the same circle. No two of the polygons share a vertex, and no three of their sides intersect at a common point. At how many points inside the circle do two of their sides intersect?

Solution. 68

Problem 6.10. (10 points) For certain real numbers a, b, and c, the polynomial $g(x) = x^3 + ax^2 + x + 10$ has three distinct roots, and each root of $g(x)$ is also a root of the polynomial $f(x) = x^4 + x^3 + bx^2 + 100x + c$. What is $f(1)$?

Solution. -7007