Olympiads: Generating Functions 2

ORMC

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Problem 0.1. Fix n. If $\sigma = (a_1, a_2, \dots, a_n)$ be a permutation of $(1, 2, \dots, n)$, an *inversion* of σ is a pair $i < j$ with $a_j < a_i$.

Find a generating function for the sequence $(a_k)_0^\infty$ where a_k is the number of permutations of $(1, \dots, n)$ with exactly k inversions.

1 Infinite Products

A lot of proofs, going back to at least Euler, make use of some kind of infinite product. Like with infinite sums, a lot of these proofs are unrigorous by modern standards, but a lot of them can be fixed with limits. If $P_1(x), P_2(x), P_3(x), \dots$ are power series, and so is $P(x)$, then we say that $\lim_n P_n(x) = P(x)$ when for every k, there is some m such that if $n > m$, the x^k term of $P_n(x)$ is the same as the x^k term of $P(x)$.

We can also define infinite products this way, by saying

$$P_1(x)P_2(x)P_3(x) \cdots = \lim_n P_1(x) \dots P_n(x)$$

whenever that limit exists.

Problem 1.1. What is the infinite product $(1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \dots$?

Problem 1.2. A *partition* of n is a list of numbers $a_1 \leq a_2 \leq \cdots \leq a_k$ with $a_1 + a_2 + \cdots + a_k = n$.

Find a generating function for the sequence $(a_n)_0^\infty$ where a_n is the number of partitions of n, expressed as an infinite product.

Problem 1.3. A partition $(a_1, \dots, a_k)$ is called *odd* when each a_i is odd, and has *distinct parts* when $a_1 < \cdots < a_k$.

Use generating functions to show that for every n, the number of odd partitions of n equals the number partitions of n with distinct parts.

2 Competition Problems

Problem 2.1 (USAMO 1996 Problem 6)**.** Determine (with proof) whether there is a subset X of the nonnegative integers with the following property: for any nonnegative integer n there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

(The original USAMO question asked about all integers, not just nonnegative - this is harder, but still approachable with generating functions.)

Problem 2.2 (IMO Shortlist 1998)**.** Let $a_0, a_1, \dots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j, k are not necessarily distinct. Determine $a_1 998$.

Problem 2.3 (USAMO 1986 Problem 5)**.** By a partition π of an integer $n \geq 1$, we mean here a representation of n as a sum of one or more positive integers where the summands must be put in nondecreasing order. (E.g., if $n = 4$, then the partitions π are $1 + 1 + 1 + 1$, $1 + 1 + 2$, $1 + 3, 2 + 2$, and 4).

For any partition π, define $A(\pi)$ to be the number of 1's which appear in π, and define $B(\pi)$ to be the number of distinct integers which appear in π (E.g., if $n = 13$ and π is the partition $1 + 1 + 2 + 2 + 2 + 5$, then $A(\pi) = 2$ and $B(\pi) = 3$).

Prove that, for any fixed n, the sum of $A(\pi)$ over all partitions of π of n is equal to the sum of $B(\pi)$ over all partitions of π of n.

Problem 2.4 (USAMO 2017 Problem 2)**.** Let $m_1, m_2, \ldots, m_n$ be a collection of n distinct positive integers. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = w_1, \ldots, w_n$ of $m_1, \ldots, m_n$, define an A-inversion of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

$$a_i \geq w_i > w_j,$$

$$w_j > a_i \geq w_i,$$

$$w_i > w_j > a_i.$$

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \ldots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \ldots, m_n$ having exactly k B-inversions.

(The original USAMO problem allowed the numbers $m_1, \ldots, m_n$ to not necessarily be distinct.)