Problem 0.1. Fix $n$. If $\sigma = (a_1, a_2, \ldots, a_n)$ be a permutation of $(1, 2, \ldots, n)$, an inversion of $\sigma$ is a pair $i < j$ with $a_j < a_i$.

Find a generating function for the sequence $(a_k)_{k=0}^{\infty}$ where $a_k$ is the number of permutations of $(1, \ldots, n)$ with exactly $k$ inversions.

1 Infinite Products

A lot of proofs, going back to at least Euler, make use of some kind of infinite product. Like with infinite sums, a lot of these proofs are unrigorous by modern standards, but a lot of them can be fixed with limits. If $P_1(x), P_2(x), P_3(x), \ldots$ are power series, and so is $P(x)$, then we say that $\lim_{n} P_n(x) = P(x)$ when for every $k$, there is some $m$ such that if $n > m$, the $x^k$ term of $P_n(x)$ is the same as the $x^k$ term of $P(x)$.

We can also define infinite products this way, by saying

$$P_1(x)P_2(x)P_3(x)\cdots = \lim_{n} P_1(x)\ldots P_n(x)$$

whenever that limit exists.

Problem 1.1. What is the infinite product $(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots$?

Problem 1.2. A partition of $n$ is a list of numbers $a_1 \leq a_2 \leq \cdots \leq a_k$ with $a_1 + a_2 + \cdots + a_k = n$.

Find a generating function for the sequence $(a_n)_{n=0}^{\infty}$ where $a_n$ is the number of partitions of $n$, expressed as an infinite product.

Problem 1.3. A partition $(a_1, \ldots, a_k)$ is called odd when each $a_i$ is odd, and has distinct parts when $a_1 < \cdots < a_k$.

Use generating functions to show that for every $n$, the number of odd partitions of $n$ equals the number partitions of $n$ with distinct parts.

2 Competition Problems

Problem 2.1 (USAMO 1996 Problem 6). Determine (with proof) whether there is a subset $X$ of the nonnegative integers with the following property: for any nonnegative integer $n$ there is exactly one solution of $a + 2b = n$ with $a, b \in X$.

(The original USAMO question asked about all integers, not just nonnegative - this is harder, but still approachable with generating functions.)

Problem 2.2 (IMO Shortlist 1998). Let $a_0, a_1, \ldots$ be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4ak$, where $i,j,k$ are not necessarily distinct. Determine $a_1$.
Problem 2.3 (USAMO 1986 Problem 5). By a partition $\pi$ of an integer $n \geq 1$, we mean here a representation of $n$ as a sum of one or more positive integers where the summands must be put in nondecreasing order. (E.g., if $n = 4$, then the partitions $\pi$ are $1 + 1 + 1 + 1$, $1 + 1 + 2$, $1 + 3$, $2 + 2$, and $4$).

For any partition $\pi$, define $A(\pi)$ to be the number of 1’s which appear in $\pi$, and define $B(\pi)$ to be the number of distinct integers which appear in $\pi$ (E.g., if $n = 13$ and $\pi$ is the partition $1 + 1 + 2 + 2 + 2 + 5$, then $A(\pi) = 2$ and $B(\pi) = 3$).

Prove that, for any fixed $n$, the sum of $A(\pi)$ over all partitions of $\pi$ of $n$ is equal to the sum of $B(\pi)$ over all partitions of $\pi$ of $n$.

Problem 2.4 (USAMO 2017 Problem 2). Let $m_1, m_2, \ldots, m_n$ be a collection of $n$ distinct positive integers. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = w_1, \ldots, w_n$ of $m_1, \ldots, m_n$, define an $A$-inversion of $w$ to be a pair of entries $w_i, w_j$ with $i < j$ for which one of the following conditions holds:

- $a_i \geq w_i > w_j$,
- $w_j > a_i \geq w_i$,
- $w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer $k$, the number of permutations of $m_1, \ldots, m_n$ having exactly $k$ $A$-inversions is equal to the number of permutations of $m_1, \ldots, m_n$ having exactly $k$ $B$-inversions.

(The original USAMO problem allowed the numbers $m_1, \ldots, m_n$ to not necessarily be distinct.)