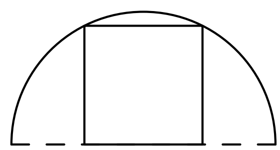
ORMC AMC 10/12 Group
Week 9: Circles

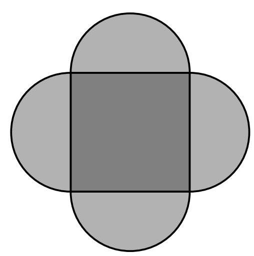
December 3, 2023

1 Warm-up Exercises

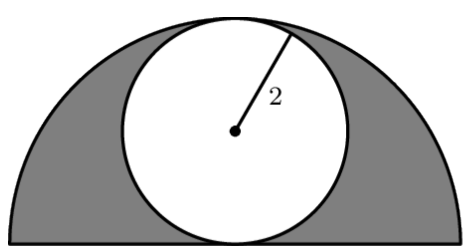
1. **(2006 AMC 10B #8)** A square of area 40 is inscribed in a semicircle as shown. What is the area of the semicircle?



2. **(2006 AMC 10B #6)** A region is bounded by semicircular arcs constructed on the side of a square whose sides measure $\frac{2}{\pi}$, as shown. What is the perimeter of this region?



3. **(2009 AMC 10A #6)** A circle of radius 2 is inscribed in a semicircle, as shown. The area inside the semicircle but outside the circle is shaded. What fraction of the semicircle's area is shaded?

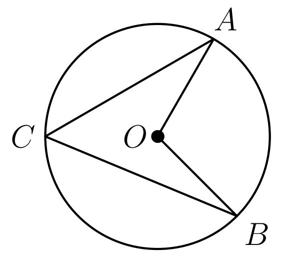


2 Theorems

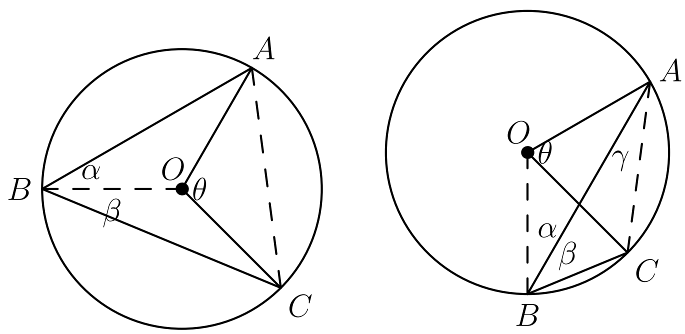
2.1 Inscribed Angle Theorem

The inscribed angle theorem tells us that the measure of an inscribed angle is half the measure of the arc that it intercepts. For example, in the diagram below, we would have

$$m\angle AOB = m\widehat{AB} = 2 \cdot m\angle ACB.$$



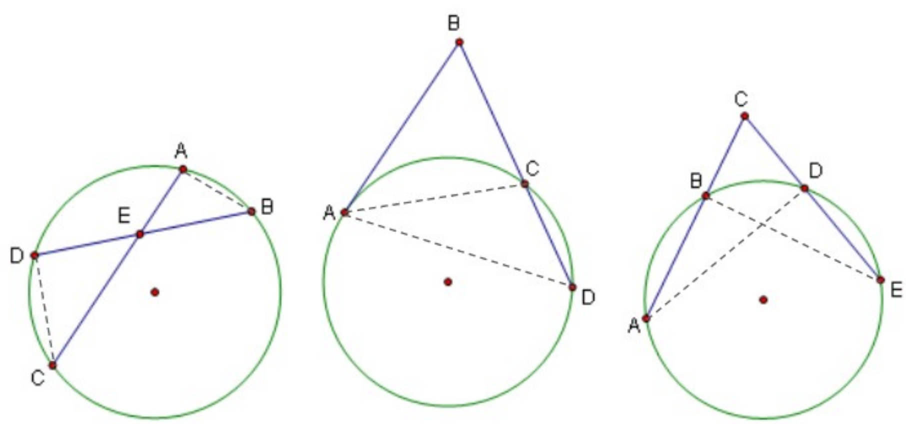
The proof of this can be broken down into two cases, where we can do some angle-chasing to find $m\angle ABC$.



Note that if a triangle has radii for two of its sides, then it is isosceles. Focusing on the central point O in the first diagram, we can see that we have $\theta = 2(\alpha + \beta)$. And, using triangles OCA and ABC in the second diagram, we have $\theta + 2\gamma + 2\alpha = \gamma + (\gamma + \alpha + \beta + \alpha) + \beta \implies \theta = 2\beta$.

2.2 Power of a Point

For a given circle, power of a point tells us about the relationships between lengths of chords, secants, and tangents that originate from the same point. The 3 main cases are given below, and all 3 can be proved using similar triangles:

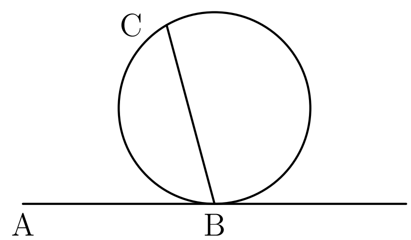


$AE \cdot CE = BE \cdot BD$ $\qquad$ $BA^2 = BC \cdot BD$ $\qquad$ $CB \cdot CA = CD \cdot CE$

3 Examples

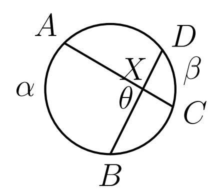
1. **(Alternate Segment Theorem)** Show that the angle between a tangent and a chord is equal to half of the measure of the arc intercepted by the angle. That is, in the diagram below, show that we have

$$m\angle ABC = \frac{1}{2} m\overset{\frown}{BC}$$



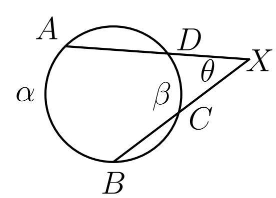
(Hint: draw a diameter that ends at B, and use the inscribed angle theorem.)

2. **(Intersecting Chords Theorem, Inside Circle)** Find the angle θ in terms of α and β.



(Hint: Use the inscribed angle theorem, and show that triangles ADX and BCX are similar.)

3. **(Intersecting Chords Theorem, Outside Circle)** Find the angle θ in terms of α and β.



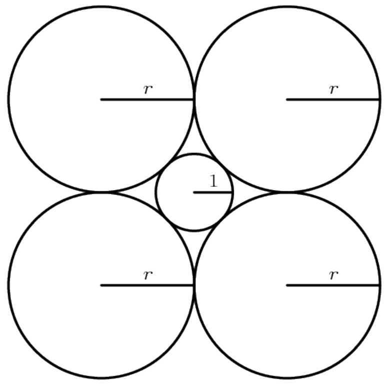
(Hint: Use the inscribed angle theorem, and show that triangles ACX and BDX are similar.)

4. **(ARML)** In a circle, chords AB and CD intersect at R. If $AR : BR = 1 : 4$ and $CR : DR = 4 : 9$, find the ratio $AB : CD$.

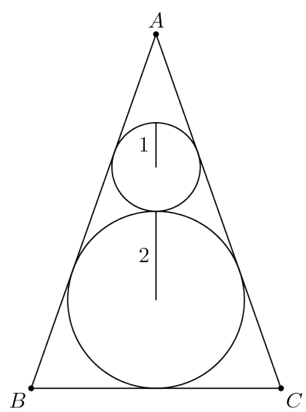
5. **(2007 AMC 10A #14)** A triangle with side lengths in the ratio $3 : 4 : 5$ is inscribed in a circle with radius 3. What is the area of the triangle?

4 Exercises

1. **(2007 AMC 10B #18)** A circle of radius 1 is surrounded by 4 circles of radius r as shown. What is r?

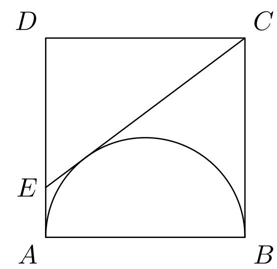


2. **(2006 AMC 10A #16)** A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of $\triangle ABC$?



3. **(2020 AMC 12B #10)** In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

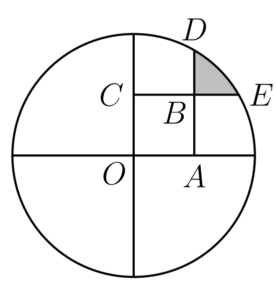
4. **(2004 AMC 12A #18)** Square $ABCD$ has side length 2. A semicircle with diameter $\overline{AB}$ is constructed inside the square, and the tangent to the semicircle from C intersects side $\overline{AD}$ at E. What is the length of $\overline{CE}$?



5. **(2020 AMC 12B #12)** Let $\overline{AB}$ be a diameter in a circle of radius $5\sqrt{2}$. Let $\overline{CD}$ be a chord in the circle that intersects $\overline{AB}$ at a point E such that $BE = 2\sqrt{5}$ and $\angle AEC = 45°$. What is $CE^2 + DE^2$?

6. **(2008 AMC 12A #13)** Points A and B lie on a circle centered at O, and $\angle AOB = 60°$. A second circle is internally tangent to the first and tangent to both $\overline{OA}$ and $\overline{OB}$. What is the ratio of the area of the smaller circle to that of the larger circle?

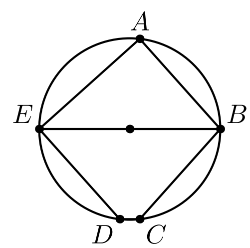
7. **(2006 AMC 10B #19)** A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides AB and CB are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by BD, BE, and the minor arc connecting D and E?



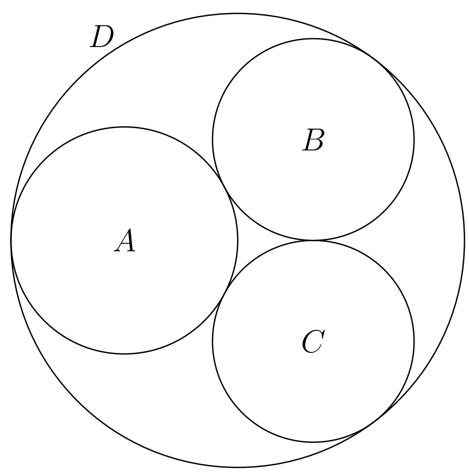
8. **(1996 AHSME #25)** Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have?

9. **(2014 AMC 12A #12)** Two circles intersect at points A and B. The minor arcs AB measure 30° on one circle and 60° on the other circle. What is the ratio of the area of the larger circle to the area of the smaller circle?

10. **(2011 AMC 10B #17)** In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?



11. **(2004 AMC 12A #19)** Circles A, B and C are externally tangent to each other, and internally tangent to circle D. Circles B and C are congruent. Circle A has radius 1 and passes through the center of D. What is the radius of circle B?



12. **(2010 AMC 10B #20)** Two circles lie outside regular hexagon $ABCDEF$. The first is tangent to $\overline{AB}$, and the second is tangent to $\overline{DE}$. Both are tangent to lines BC and FA. What is the ratio of the area of the second circle to that of the first circle?

13. **(2013 AMC 10A #23)** In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

14. **(2004 AMC 12A #22)** Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?

15. **(2019 AMC 10B #23)** Points $A = (6, 13)$ and $B = (12, 11)$ lie on circle ω in the plane. Suppose that the tangent lines to ω at A and B intersect at a point on the x-axis. What is the area of ω?

Challenge Exercises

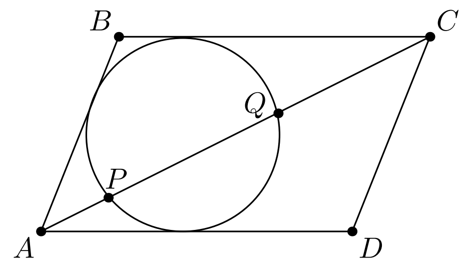
1. (Complex Power of a Point)

   (a) In the complex plane, we can represent a line as the set of points $\{z + wt : t \in \mathbb{R}\}$, for fixed complex numbers z, w. In this case, z is a point that the line passes through, w is the direction of the line, and $\left|\frac{t}{w}\right|$ is the distance away from z along the line.
   Find the complex-plane equation for the line $y = \frac{3}{4}x - 3$, and find the length of the segment between $(0, -3)$ and $(8, 3)$.

   (b) We can represent a circle centered at the origin in the complex plane by the equation $z\overline{z} = r^2$ (which comes from squaring the equation $|z| = r$). Find an equation for the values of t where the line $z_0 + wt$ intersects the circle of radius 1 centered at the origin.

   (c) Using the equation from part (b), there should be two points of intersection, call them z_+ and z_-. What is the product of the lengths $\overline{z_0 z_+}$ and $\overline{z_0 z_-}$?

2. **(2022 AIME I #11)** Let $ABCD$ be a parallelogram with $\angle BAD < 90°$. A circle tangent to sides $\overline{DA}$, $\overline{AB}$, and $\overline{BC}$ intersects diagonal $\overline{AC}$ at points P and Q with $AP < AQ$, as shown. Suppose that $AP = 3$, $PQ = 9$, and $QC = 16$. Then the area of $ABCD$ can be expressed in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

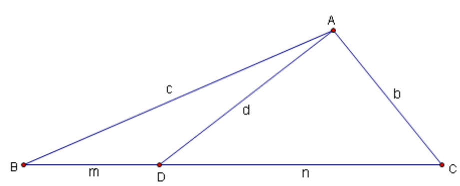


3. **(2019 AIME I #6)** In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

   (Hint: Consider the circles with diameter KN and ON)

4. **(2005 AIME I #15)** In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

   (Hint: In addition to Power of a Point, you may need to use Stewart's Theorem:



$$a = m + n, \qquad amn + ad^2 = mb^2 + nc^2$$

   Stewart's theorem can be derived using law of cosines, and dropping an altitude from A to $\overline{BC}$.)