Pigeons, Pigeons Everywhere!

Math Circle (Intermediate)

October 28, 2012

SPECIAL DIRECTIONS: For any problem that asks you to "explain" something, you MUST write a complete sentence.

The pigeonhole principle can help us solve a variety of geometry problems.

1. What is the largest number of kings that can be placed on a chessboard so that no two of them can put each other in check? Justify your answer.

2. Can an equilateral triangle be covered completely by two smaller equilateral triangles? **Explain.**

3. **Prove or disprove:** Among any five points located inside or on the boundary of a unit square there are two points at most $\frac{1}{\sqrt{2}}$ distance apart.

4. 51 points are scattered inside a square with side length 1 meter.

   (a) Is it always true that three of these 51 points can be covered by a square with side 10 centimeters?

	Item	Quantity
Holes		
Pigeons		

   (b) Is it always true that three of these 51 points can be covered by a square with side 25 centimeters?

	Item	Quantity
Holes		
Pigeons		

   (c) What is the minimum length of a side of a square that will always be able to cover some set of three of these 51 points?

5. **Playing with dart boards.**

   (a) Seven darts are thrown onto a circular dart board of radius 10 inches. **Prove or disprove:** There must be two darts which are at most 10 inches apart.

(b) Six darts are thrown onto a circular dart board of radius 10 inches. **Prove or disprove:** There must be two darts which are at most 10 inches apart.

(c) Five darts are thrown onto a circular dart board of radius 10 inches. **Prove or disprove:** There must be two darts which are at most 10 inches apart.

6. An equilateral triangle ABC and a square MNPQ are inscribed in a circle of circumference S. None of the vertices of the triangle coincide with a vertex of the square. Thus, their vertices divide the circle into seven arcs. **Prove or disprove**: At least one of the arcs must not be longer than $\frac{S}{24}$.

The pigeonhole principle can also help us solve problems in divisibility and number theory.

7. Use the pigeonhole principle to prove that the decimal expansion of a rational number $\frac{m}{n}$ is eventually repeating. (Note that if a decimal "ends", like 0.75, it can still be considered repeating: .750000000...).

8. Suppose you are asked to convert a rational number $\frac{m}{n}$ to a decimal. What is the largest number of decimal places you would possibly have to write of the decimal representation before you could determine the period at which it repeats? **Prove it**! (Use the pigeonhole principle.)

9. **Prove or disprove**: There is a number consisting entirely of ones that is divisible by 7777.

	Item	Quantity
Holes		
Pigeons		

10. **Prove or disprove:** There do not exist two powers of 2 that differ by a multiple of 2013.

11. A 10 by 10 table is filled in with positive integers so that adjacent integers (i.e., integers are adjacent if their squares share a side) differ by 5 or less. **Prove or disprove**: The table must contain two identical integers. (Hint: If a rook on a 10x10 chessboard takes the shortest available path from square A to square B, what is the largest number of squares the rook could possibly have to cover to get from square A to square B?)

12. Fourteen Math Circle students ate 100 pieces of Halloween candy.
Prove or disprove: Of these fourteen Math Circle students, there must be some pair that ate the same number of pieces of candy.

13. Sony randomly selects eight positive integers, all less than 15.

    (a) How many pairs can be formed from these eight integers?

    (b) **Prove or disprove**: There must be three pairs that have the same positive difference.

14. Of 100 people seated at a round table, more than half are men. Is it true that there must be two men who are seated diametrically opposite each other? **Explain**, making explicit reference to the pigeonhole principle.

15. Each box in a 3×3 arrangement of boxes is filled with one of the numbers: -1; 0; 1. Prove that of the possible sums along the rows, the columns, and the diagonals, two sums must be equal.

16. Six distinct positive integers are randomly chosen between 1 and 2012, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

17. Prove that among a group of six people there are either

    - three people who all know each other,
      OR
    - three people who are complete strangers to each other.

    (Hint: Pick one person at random to study.)[1]

[1] Some problems are taken from:
D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"
D. Patrick "Introduction to Counting and Probability"