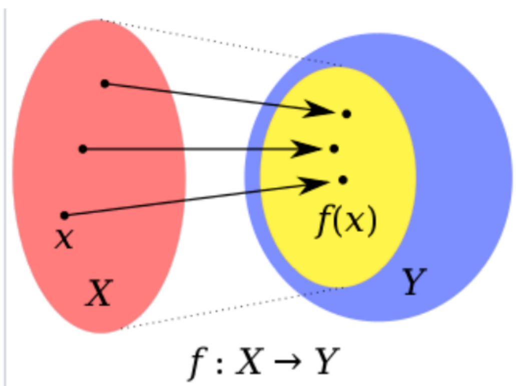
Intermediate 1 **Dec 3, 2023**

Functions and Cardinality

1 Functions

Let X, Y be sets. A **function** from X to Y is an assignment to each element in X an element in Y. In other words, a function is a way to move points from the set X to points in another set Y.



Let f be the function moving points from a set X to a set Y. Then we write $f : X \to Y$. This is read as "f is a function from X into Y". Sometimes we use the word "map" instead of "function".

The set X is called the **domain** and the set Y is called the **codomain**.

To show how the function moves points, we usually write the function explicitly. If $x \in X$ and $y \in Y$, we write $f(x) = y$. We call x the *argument* of the function, and call y the *value* of $f(x)$.

For example, Let $f : \mathbb{N} \to \mathbb{N}$, and $f(x) = 2x$. What this function does is it takes any natural number, and doubles it. So we have

$$f(0) = 0, \qquad f(1) = 2, \qquad f(2) = 4, \qquad \dots$$

Here is another example: Let $f : \mathbb{N}\ \{0, 1, 2\} \to S$, where S is the set of regular shapes. What f will do is take a number, and send that number to the shape with that many vertices. For example,

$$f(3) = \triangle, \qquad f(4) = \Box, \qquad \dots$$

Problem 1.1. *Let C be the set of countries, W be the set of fictional writers, and M be the set of mathematicians. Consider the function $f : C \to W \times M$, where f takes in a country, and outputs an ordered pair of the most famous fictional writer and mathematician born in that country. For example,*

$$f(\textit{USA}) = (\textit{Ernest Hemingway, Michael Freedman})$$

What do you think f(Russia) is? How about f(Scotland)?

Problem 1.2. *Give 2 examples of functions in our daily life.*

Problem 1.3. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function $f(x) = x^2$. What is $f(1), f(2)$, and $f(-3)$?*

Does $f(\pi)$ exists? Why?

Let $f : X \to Y$ be a function. Let A be a subset of our domain, $A \subset X$. Then the **image** of A is the set of everything A gets sent to. This is written as $f(A)$. Using set notation, we have

$$f(A) = \{f(x) \in Y \ : \ x \in X\}.$$

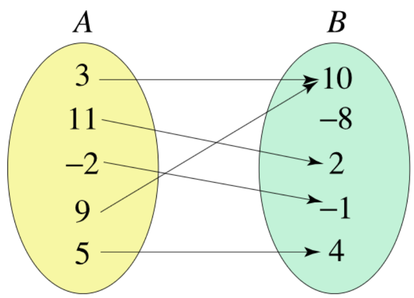
We call $f(X)$, the image of the entire domain, as *the image of f*.

For example, let $A = \{1, 2, 3\}$. The function $f : A \to \mathbb{Z}$ given by $f(x) = 3x$ has the image

$$f(A) = \{3, 6, 9\}.$$

Note that by definition, the image of a function is always a *subset* of the codomain, but not necessarily the entire codomain.

Problem 1.4. *Consider the function $f : A \to B$ given by the diagram*



- *What is the domain of f?*

- *What is the codomain of f?*

- *What is the image of f?*

Problem 1.5. *What is $f(\emptyset)$?*

Problem 1.6. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function $f(x) = |x|$. What is the image of the function?*

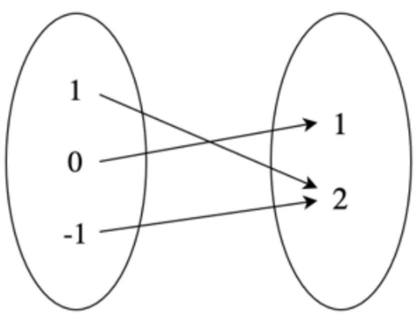
Is the image the same as the codomain?

A function f is called **surjective**, or we say f is a surjection, if the image of the domain is equal to the codomain;

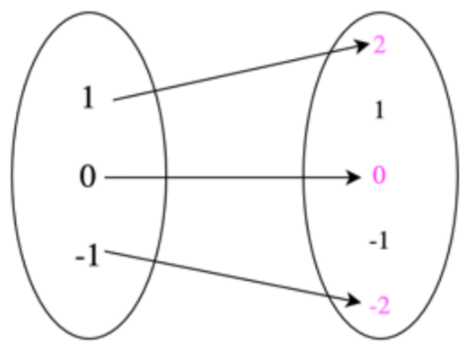
$$f(X) = Y.$$

Another way to say this is to say that f is an onto function. In this case, instead of saying "f is a map from X into Y ", we would say "f is a map from X onto Y ".

For example, for the two functions below, the function on the left is surjective, while the function on the right is not.



Surjective



Not surjective

Problem 1.7. *Which of the function(s) below are surjective? Why?*

(a) $f : \mathbb{Z} \to \mathbb{Z}$ *by* $f(x) = x^2$.

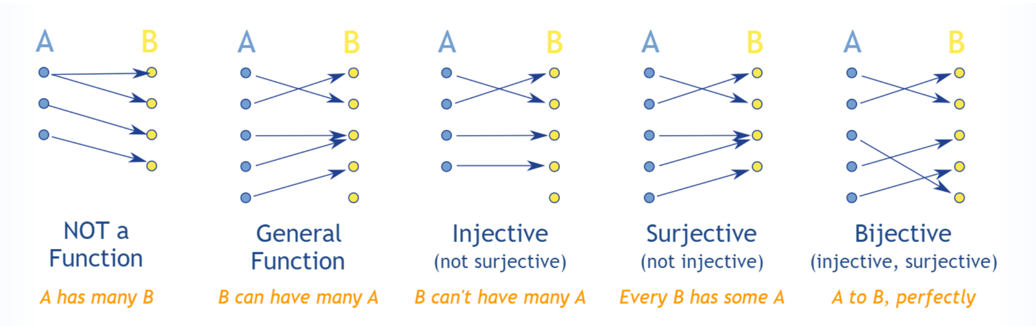
(b) $f : \mathbb{Z} \to \mathbb{N}$ *by* $f(x) = x^2$.

(c) $f : \mathbb{R} \to \mathbb{Z}$ *by* $f(x) = \lfloor x \rfloor$, *where* $\lfloor x \rfloor$ *means the biggest integer not greater than* x.

A function f is called **injective**, or we say f is an injection, if no two points in X gets sent to the same point in Y.

To prove a function is injective, assume $f(a) = f(b)$, then prove that $a = b$. By contrapositive, this is the same as saying if $a \neq b$, then $f(a) \neq f(b)$. Another way to say this is to say that f is a *one-to-one* function, or that f is one-to-one.

A function f is called **bijective**, or we say f is a bijection, if f is both injective and surjective. Another way to say this is to say that f is a *one-to-one correspondence*. That is, f takes points from X, and maps it to points to Y , such that each point in X is sent to a unique point in Y , and every point in Y is the image of some point in X.



Problem 1.8. *In our real life, give an example of a function that is injective but not surjective. Then give an example that is surjective but not injective.*

Problem 1.9. *Give a subset X of $\mathbb{R}$ such that the function $f : X \to \mathbb{R}$ by $f(x) = x^2$ is injective. Try to make your subset X as big as possible.*

Problem 1.10. *Let $f : X \to Y$ be a function, and let $A, B \subset X$. Prove that*

$$f(A \cup B) = f(A) \cup f(B).$$

Problem 1.11. *Suppose $f : X \to Y$ is injective, and $A, B \subset X$. Prove that*

$$f(A \cap B) = f(A) \cap f(B).$$

Problem 1.12. *Give an example of f not injective such that $f(A \cap B) \neq f(A) \cap f(B)$.*

Problem 1.13. *Suppose* $f : X \to Y$ *is injective. Show that there is a surjective function* $g : Y \to X$*.*

Problem 1.14. *Suppose* $f : X \to Y$ *is surjective. Show that there is an injective function* $g : Y \to X$*.*

Problem 1.15. *Suppose* $f : X \to Y$ *is bijective. Show that there is a bijective function* $g : Y \to X$*.*

Problem 1.16. *Find a bijective function* $f : \mathbb{N} \to \mathbb{Z}$*.*

2 Cardinality

Intuitively, the cardinality of a set is how "big" it is. If A is a set, we let $|A|$ denote its cardinality, read "the cardinality of A". We have previously discussed the cardinality of finite sets, where we said the cardinality of a finite set is the number of elements in it. Notice that when we count the elements in a set, we are just assigning the numbers $1, 2, \ldots$ to the elements in the set. Let us formalize this notion:

Let $\mathbb{N}_n$ be the first n counting numbers, i.e. $\mathbb{N}_n = \{1, \ldots, n\}$. We define $|\mathbb{N}_n| = n$. We also say that a set A has **cardinality** n, or $|A| = n$, if there is a bijection between $\mathbb{N}_n$ and A. We will use this definition of cardinality for the rest of the handout. Fact: if there is a bijection between A and $\mathbb{N}_n$ for some n, then there is no bijection between A and $\mathbb{N}_m$ for any $m \neq n$. That is, the cardinality of a finite set is unique.

Problem 2.1. *Let A and B be finite sets. Prove that $|A| = |B|$ if and only if there is a bijection between A and B.*

Problem 2.2. *Let A and B be finite sets. Prove that $|A| \leq |B|$ if and only if there is an injection from A to B.*

Problem 2.3. *Let A and B be finite sets. Prove that $|A| \geq |B|$ if and only if there is a surjection from A to B.*

Now, let us generalize to infinite sets. A set A is called **infinite** if it is not finite. That is, there is no bijection from A to $\mathbb{N}_n$ for any n. Let A and B be sets (finite or infinite). We write:

- $|A| \leq |B|$ if there is an injection from A to B.
- $|A| < |B|$ if there is an injection from A to B, but no bijection.
- $|A| \geq |B|$ if there is a surjection from A to B.
- $|A| > |B|$ if there is a surjection from A to B, but no bijection.

There are two types of infinities we care about: countable infinity and uncountable infinity.

A set A is called **countably infinite** if there is a bijection from A to $\mathbb{N}$. The name countable comes from the fact that $\mathbb{N}$ is the set of counting numbers. That is, if A is countable, then it is possible to "count" the elements of A. We define $|\mathbb{N}| = \aleph_0$.

Problem 2.4. *Show that $\mathbb{N}$ is the "smallest" infinite set. That is, given an infinite set A, construct a surjection from A to $\mathbb{N}$.*

Problem 2.5. *Show that* $|\mathbb{N}| = |\mathbb{Z}|$.

Problem 2.6. *Show that* $|\mathbb{N}| = |\mathbb{N} \times |\mathbb{N}|$.

Problem 2.7. *Show that* $|\mathbb{N}| = |\mathbb{Q}|$.

A set A is said to be **uncountably infinite**, or uncountable, if A is infinite but not countable.

Problem 2.8 (Challenge)**.** *Prove that* $|\mathbb{R}| > |\mathbb{N}|$