Geometry

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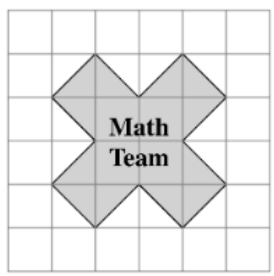
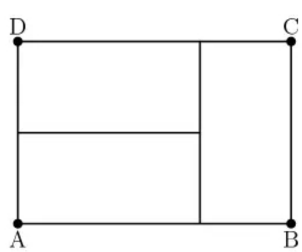
1 Warm Up

The Math Team designed a logo shaped like a multiplication symbol, shown below on a grid of 1-inch squares. What is the area of the logo in square inches?



(A) 10 (B) 12 (C) 13 (D) 14 (E) 15

Three identical rectangles are put together to form rectangle ABCD, as shown in the figure below. Given that the length of the shorter side of each of the smaller rectangles is 5 feet, what is the area in square feet of rectangle ABCD?



(A) 45 (B) 75 (C) 100 (D) 125 (E) 150

2 Introduction

Today, we'll be going over another highly present topic throughout all of the AMC tests, whether it be 8, 10, 12, or even AIME and USAMO: geometry. It might seem daunting at first, but the concepts themselves are rooted in common sense topics that we are already familiar with. Let's start with one very important question...

2.1 What is Geometry?

Geometry is a composite word. In ancient Greek, *geo* meant 'earth' and *metria* meant 'measure.' Geometry was born as a humble art of measuring land lots. It progressed to become the part of mathematics responsible for visual understanding of mathematical objects. For example, geometry shows that gravity is a manifestation of curvature of warped spacetime in the Minkowski space, a 4-dimesional space equipped with a non-Euclidean way to measure distances between points, called the Minkowski metric.

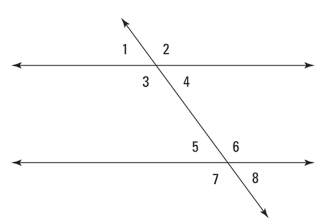
3 Background

3.1 Congruence

Two geometric objects are called congruent, if there exists a distance-preserving motion of the space the objects belong to that makes one object coincide with the other point-wise. A motion is either a translation or a rotation. Reflections in points, lines, and planes preserve distances as well, but can change orientation.

The symbol that is normally used to signify congruence is $\cong$.

3.2 Lines and Angles



When two lines intersect, four angles are formed. Angles 1 and 2 in the figure above are called *linear angles*. Linear angles are **supplementary**, meaning the sum of their measures is 180 degrees. Angles 1 and 4 above are called *vertical angles*. Vertical angles are congruent.

Parallel Lines, like the ones in the picture above are in the same plane but never intersect. When a line intersects a pair of parallel lines, though, it is called a *transversal*. Three types of angles are created by transversal lines, as seen above:

Angles 1 and 5 are Corresponding Angles (CA).

(1) Name three other pairs of corresponding angles.

Angles 5 and 4 are Alternate Interior Angles (AIA). They are called interior because they are between the parallel lines, and alternate because they are on opposite sides of the transversal. Angles 3 and 6 are also alternate interior angles.

(2) Name one other pair of alternate interior angles.

Angles 1 and 8 are Alternate Exterior Angles (AEA).

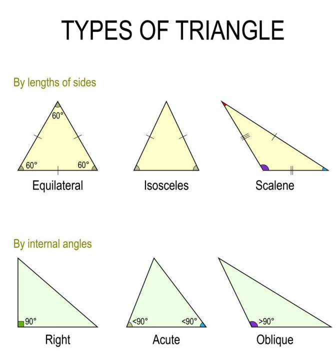
Exterior because they are outside the parallel lines, and alternate because they are on opposite sides of the transversal. Angles 2 and 7 are also alternate exterior angles.

(3) Name one other pair of alternate exterior angles.

All three of these types of angles above form congruent pairs.

4 Triangles

4.1 Triangle Basics



Triangles are classified by their sides and their angles, as seen above.

A triangle with three congruent sides is called **equilateral**. A triangle with two or more congruent sides is called **isosceles**. A triangle with no congruent sides is called **scalene**.

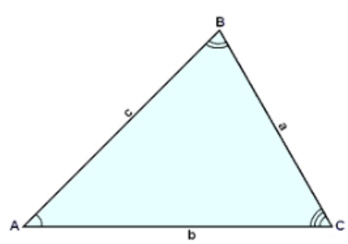
A triangle in which all of the angles are less than 90° is called **acute**. A triangle in which one angle is 90° is called a **right** triangle. A triangle in which one angle measure is greater than 90° is called **oblique** or **obtuse**.

4.2 Triangle Angle and Side Properties

In any triangle, the largest angle will always be opposite the longest side, and the smallest angle will be opposite the shortest side. The marks seen above are used to indicate congruent sides and a right angle.

Triangle Angle Sum: The interior angles of a triangle will always add up to 180°! This is helpful when you know two angles of a triangle and need to find the third.

The Triangle Inequality: There is a property of triangles which seems almost too obvious to mention, but many competition problems have use problems that cause students to overlooked the following:



The sum of the length of any two sides of a triangle must be greater than the length of the third side:

$$a + b > c$$

Consequently, the difference between the lengths of any two sides of a triangle must be less than the length of the third side:

$$a > c - b$$

4.3 The Pythagorean Theorem

Perhaps the most useful (and most often used) tool in Geometry is the Pythagorean theorem. The sum of the squares of the legs *of a right triangle* is equal to the square of the hypotenuse, or, as you probably know it:

$$a^2 + b^2 = c^2$$

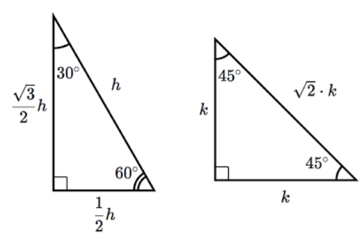
Knowing the most common Pythagorean triples is very helpful in recognizing and solving problems. Can you name some common Pythagorean triples?

There are some patterns which also make many of the Pythagorean triples easy to remember. Of course, the easiest triples to remember are multiples of the common ones like 3-4-5, 6-8-10, and 15-20-25.

4.4 Special Right Triangles

There are two right triangles which occur quite frequently. Knowing the relationship between their sides is key to solving many competition problems which involve the Pythagorean theorem.

So what exactly are the special right triangles?



The 30-60-90 right triangle (or one-half of an equilateral triangle):
The 45-45-90 right triangle (or one-half of a square) where given leg length a, the hypotenuse is $a\sqrt{2}$.

4.5 Triangle Area

It is really important to know the area of a triangle! The most common formula that many of you may know already is

$A = \frac{1}{2}b * h$

where A is the area, b is the base, and h is the height of the triangle. However, there is one other formula that is also really helpful on the AMC exam!

It is called *Heron's Formula*, and the entire formula is based only off of SIDE LENGTHS. It is as follows:

$$A = \sqrt{(s)(s-a)(s-b)(s-c)}$$

where A is the area of the triangle, a, b, and c are the side lengths of the triangle, and s is the *semi-perimeter* of the triangle. The semi-perimeter of a triangle is defined as

$$s = \frac{a+b+c}{2}$$

4.6 Similar Triangles

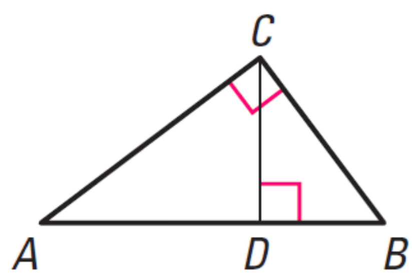
Polygons are described as similar if all corresponding angles are congruent, and all corresponding sides are proportional. In simplest terms, similar polygons are the same shape but not necessarily the same size. Polygons may be enlarged, rotated, or even reflected and remain similar to the original.

Similarity is also indicated by the symbol $\sim$, as in $ABC \sim DEF$.

Most similarity problems involve triangles. Any two triangles which share the same angle measures are similar (AAA), or if they share one angle and have two adjacent proportional sides (SAS).

The ratio of the length of the sides of similar figures is called **scale factor**. Multiplying all the sides of one figure by the scale factor will achieve all the sides of another figure, and the opposite is true with division.

Similar Right Triangles



The altitude to the hypotenuse of a right triangle divides it into two smaller right triangles which are similar to the original. Given any two of the five segments in the right triangles above, all of the remaining lengths can be found using a combination of similarities and the Pythagorean theorem. Let's try it out!

(1) The altitude to the hypotenuse of a right triangle divides the hypotenuse into segments of length 6cm and 10cm. Find the area of the right triangle.

5 Polygons

Next, we'll be looking at shapes with more than just three sides: polygons!

5.1 Polygons Basics

What is a polygon?

A polygon is a closed plane figure. Diagonals in a polygon connect non-adjacent pairs of vertices. Polygons may be convex or concave. In a convex polygon, all of the diagonals are contained entirely within the polygon. Or, in other words, all the angle measures are less than 180° in a convex polygon. A polygon that is not convex is concave.

There are also special polygons. An **equilateral** polygon has all sides of equal length. An **equiangular** polygon has (you guessed it) all angles of equal measure. A regular polygon is equilateral and equiangular, but an equilateral or equiangular polygon is not always necessarily both.

Polygon Angle Sum If a polygon has n sides, we can always divide it into $(n-2)$ triangles by drawing $(n-3)$ diagonals. The sum of the angle measures in each triangle is 180 degrees, so the sum of the angles in a polygon with n sides is $180(n-2)$.

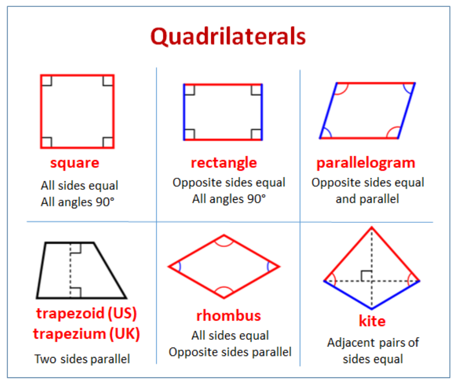
Perhaps even more useful is the sum of the exterior angles in a polygon. In any polygon, the sum of its exterior angles will always equal 360°.

5.2 Polygon Types

Polygons with different numbers of side lengths have different names. A three sided polygon, as we saw, is called a triangle. A four sided polygon is called a quadrilateral. A five sided polygon is called a pentagon. Other notable polygons are:
Hexagon (6)
Heptagon or Septagon (7)

Octagon (8)
Nonagon (9)
Decagon (10)
Dodecagon (12)

There are also a couple quadrilaterals that are worth noting for the AMC.



Trapezoids have exactly one pair of parallel sides, called its bases. A midsegment of a trapezoid connects the midpoints of the non-parallel sides. The midsegment will always be parallel to the bases, and its length is the average of the bases. An isosceles trapezoid has congruent non-parallel sides and congruent base angles.

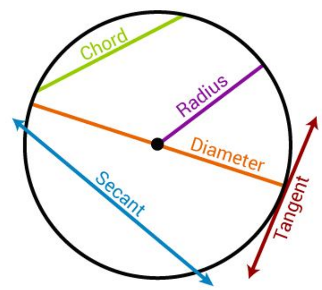
Parallelograms have two pairs of parallel sides. The parallel sides are congruent. Diagonals of a parallelogram bisect each other. If a parallelogram is equilateral, it is called a rhombus. The diagonals of a rhombus have the additional property that they are perpendicular to each other. If a parallelogram is equiangular (90 degrees for each angle) it is called a rectangle. A regular quadrilateral is a square.

A kite is a quadrilateral with to distinct pairs of congruent sides which are adjacent. The angles where congruent sides meet are called vertex angles. The angles where non-congruent sides meet are called non-vertex angles. Non-vertex angles are congruent. The diagonals of a kite are perpendicular. The diagonal connecting the vertex angles bisects the other diagonal as well as both vertex angles.

6 Circles

Now, on to circles! We can first start with some vocabulary (there is a lot!).

6.1 Circles Basics

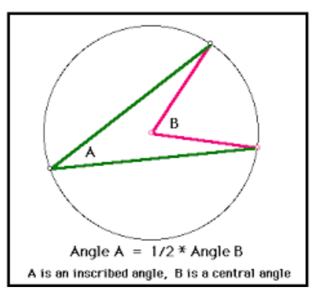


Aside from the more commonly known parts of a circle (radius, circumference, and diameter), you should also know the terms chord, secant, and tangent. As with many terms in geometry, these are easiest to define with a diagram:

The green line segment is a **chord**. The endpoints of a chord lie on the circle. A **diameter** (the orange line) is a chord which passes through the center of a circle.

The blue line is a **secant**. A secant is a line which passes through the circle.

The red line is a **tangent**. A tangent is a line which intersects a circle at exactly one point. Note that tangent can also be used as an adjective, describing two figures which touch at one point.



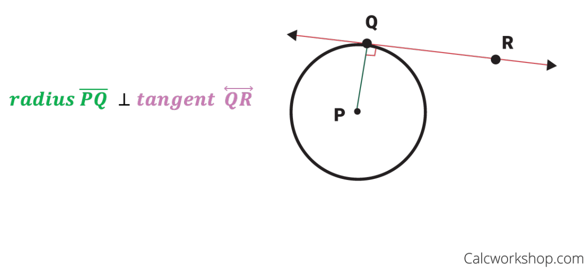
An **arc** is a portion of a circle's circumference. Arcs are measured in degrees and the measure of an are is the same as the corresponding central angle (we will go over these more in a following section!)

An inscribed angle is an angle with its vertex and endpoints on the circumference of a circle. In the above diagram, angle A is an inscribed angle, while the angle corresponding to its arc length is angle B.

6.2 Circle Properties

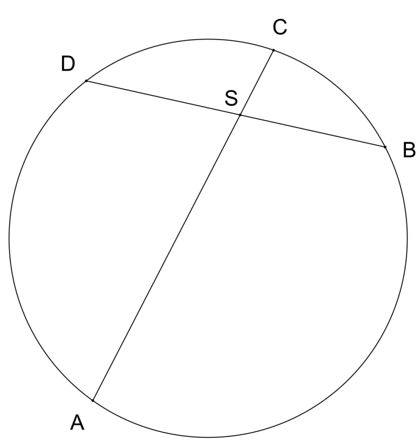
There are a couple highly notable circle properties we'll be covering today. If you're interested, you can also check out other properties as well, such as power of a point!

1) A tangent will be perpendicular to the radius drawn to the point of tangency, as shown below.

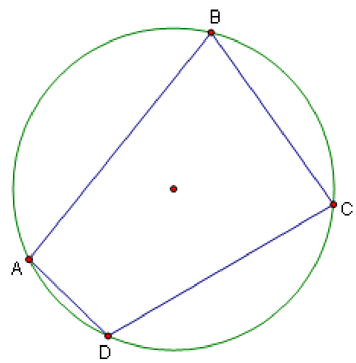


2) The measure of an inscribed angle is equal to half the measure of the intercepted arc.
See the picture from the section above!

3) The products of the line segments of created by intersecting chords are equal. So in this graphic, $(SA)(SC) = (SD)(SB)$.

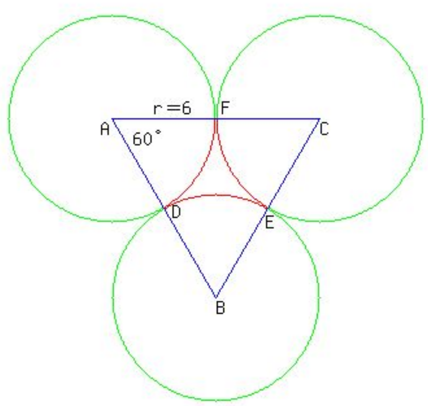


4) Opposite angles in cyclic quadrilaterals, or quadrilaterals inscribed in a circle, are supplementary: or in other words, they add up to 180, as shown below.



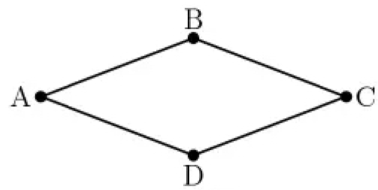
5) The circumference of a circle is $\pi * d$ or $2\pi * r$, and the arc length of a specific angle is the ratio of that angle measure to 360 times the circumference. Or in other words, $\theta/360 * C$ where θ is the angle measure of the arc and c is the circumference of the circle.

Let's do a problem to practice all of what we have learned here: **(1) What is the perimeter of the GREEN outline below which consists of three congruent tangent circles of radius 6cm?**



7 Practice!

Quadrilateral ABCD is a rhombus with perimeter 52 meters. The length of diagonal AC is 24 meters. What is the area in square meters of rhombus ABCD?

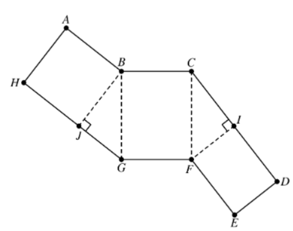


$(A)60$ $(B)90$ $(C)105$ $(D)120$ $(E)144$

Bob is tiling the floor of his 12 foot by 16 foot living room. He plans to place one-foot by one-foot square tiles to form a border along the edges of the room and to fill in the rest of the floor with two-foot by two-foot square tiles. How many tiles will he use?

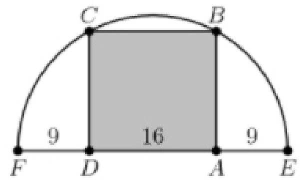
$(A)48$ $(B)87$ $(C)91$ $(D)96$ $(E)120$

The figure below shows a polygon ABCDEFGH, consisting of rectangles and right triangles. When cut out and folded on the dotted lines, the polygon forms a triangular prism. Suppose that AH=EF=8 and GH=14. What is the volume of the prism?



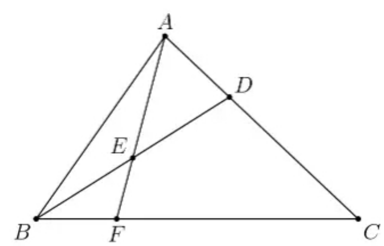
$(A)112$ $(B)128$ $(C)192$ $(D)240$ $(E)288$

Rectangle ABCD is inscribed in a semicircle with diameter FE as shown in the figure. Let DA = 16, and let FD = AE = 9.What is the area of ABCD?



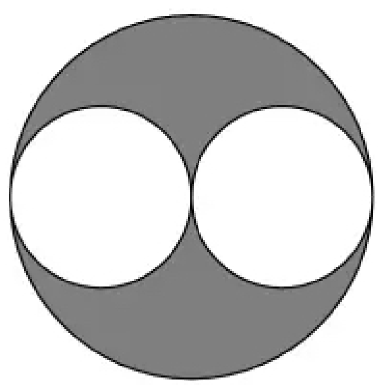
$(A)240$ $(B)248$ $(C)256$ $(D)264$ $(E)272$

In triangle ABC, point D divides side AC so that AD:DC=1:2. Let E be the midpoint of BD and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



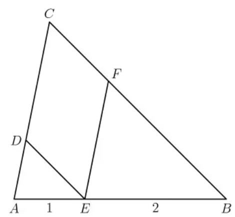
$(A)24$ $(B)30$ $(C)32$ $(D)36$ $(E)40$

In the diagram below, a diameter of each of the two smaller circles is a radius of the larger circle. If the two smaller circles have a combined area of 1 square unit, then what is the area of the shaded region, in square units?



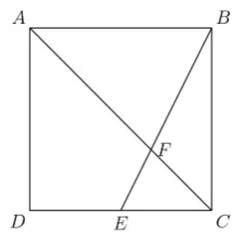
$(A)\frac{1}{4}$ $(B)\frac{1}{3}$ $(C)\frac{1}{2}$ $(D)1$ $(E)\frac{\pi}{2}$

In $\triangle ABC$, a point E is on $\overline{AB}$ with AE=1 and EB=2. Point D is on $\overline{AC}$ so that $\overline{DE} \parallel \overline{BC}$ and point F is on $\overline{BC}$ so that $\overline{EF} \parallel \overline{AC}$. What is the ratio of the area of CDEF to the area of $\triangle ABC$?



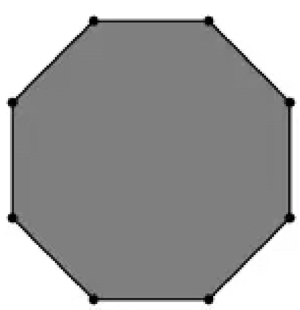
$(A)\frac{4}{9}$ $(B)\frac{1}{2}$ $(C)\frac{5}{9}$ $(D)\frac{3}{5}$ $(E)\frac{2}{3}$

Point E is the midpoint of side $\overline{CD}$ in square ABCD, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral AFED is 45. What is the area of ABCD?



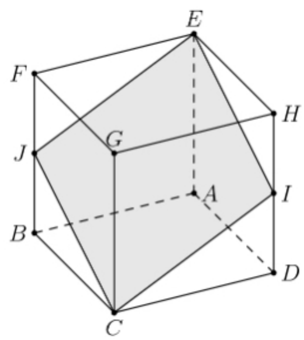
$(A)100$ $(B)108$ $(C)120$ $(D)135$ $(E)144$

From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?



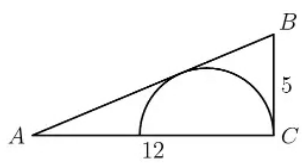
$(A)\frac{2}{7}$ $(B)\frac{5}{42}$ $(C)\frac{11}{14}$ $(D)\frac{5}{7}$ $(E)\frac{6}{7}$

In the cube ABCDEFGH with opposite vertices C and E, J and I are the midpoints of edges $\overline{FB}$ and $\overline{HD}$, respectively. Let R be the ratio of the area of the cross-section EJCI to the area of one of the faces of the cube. What is R^2?



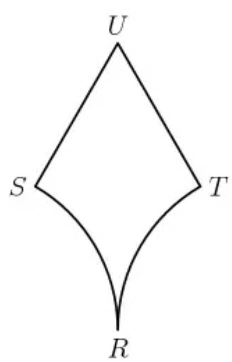
$(A)\frac{5}{4}$ $(B)\frac{4}{3}$ $(C)\frac{3}{2}$ $(D)\frac{25}{16}$ $(E)\frac{9}{4}$

In the right triangle ABC, AC=12, BC=5, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?



$(A)\frac{7}{6}$ $(B)\frac{13}{5}$ $(C)\frac{59}{18}$ $(D)\frac{10}{3}$ $(E)\frac{60}{13}$

In the figure shown, $\overline{US}$ and $\overline{UT}$ are line segments each of length 2, and $m\angle TUS = 60°$. Arcs TR and SR are each one-sixth of a circle with radius 2. What is the area of the region shown?



$(A)3\sqrt{3}-\pi$ $(B)4\sqrt{3}-\frac{4\pi}{3}$ $(C)2\sqrt{3}$ $(D)4\sqrt{3}-\frac{2\pi}{3}$ $(E)4+\frac{4\pi}{3}$