LAMC Junior Circle **October 28, 2012**

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Prepare for the Halloween!

Problem 1 *Find all the solutions of the following cryptarithm.*

$$
\begin{array}{ccccccc}
 & & G & H & O & S & T \\
+ & & G & H & O & S & T \\
\hline
 & & H & O & U & S & E
\end{array}
$$

Extra problems for those who don't need to go over parts of the previous handout.

Problem 2 *The first digit of a three-digit number is 7. The second three-digit number is obtained from the first by moving the 7 from the beginning to the end of the number. The resulting number is 117 less than the original number. What is the second number?*

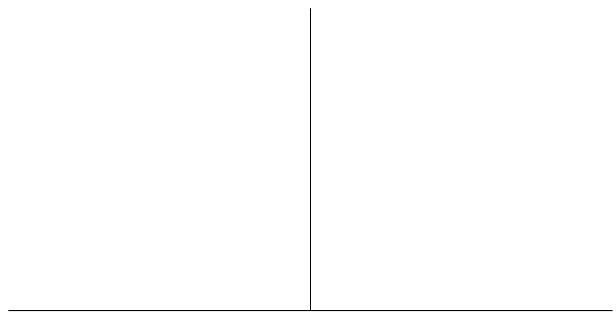
Problem 3 *A flight from Moscow, Idaho to New York City departs Moscow at noon and lands in NYC at 8:00 PM. On the way back, the plane leaves NYC at midnight and lands in Moscow, ID at 4:00 AM. How long is the flight?*

Problem 4 *Find all the positive two-digit numbers divisible by both of their own digits.*

Problem 5 *A floor lamp has five light bulbs and a six-position switch. Each position of the switch lights a different number of bulbs, zero through five. Some of the bulbs have burnt out and light up no more. Can a person unfamiliar with the workings of the switch figure out what bulbs need to be replaced?*

Distances between points in the Euclidean plane

Definition 1 *An angle that is a half of a straight angle is called the right angle.*



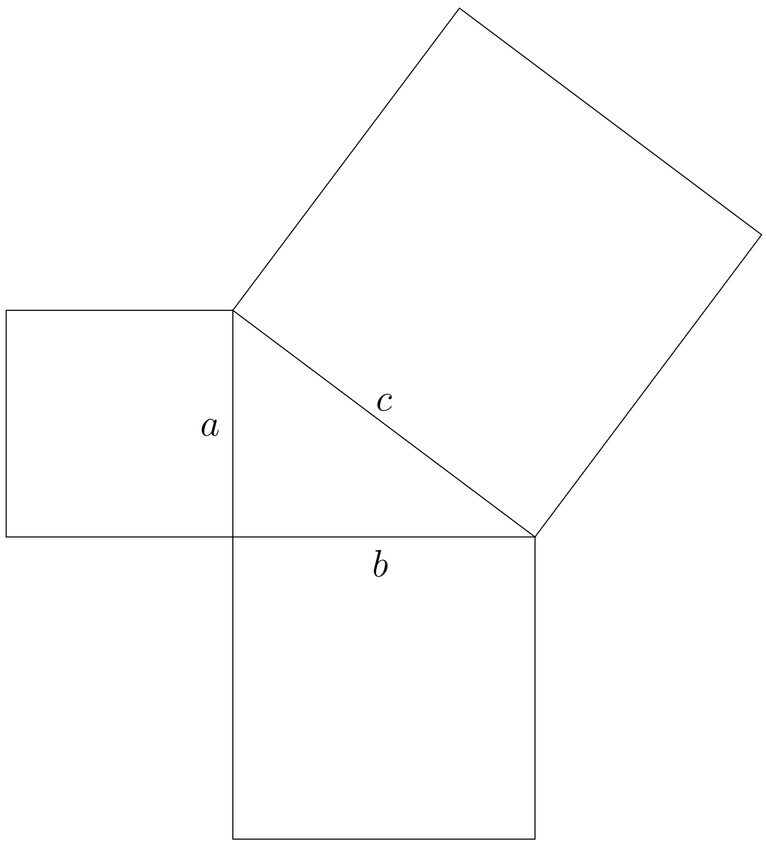
Problem 6 *Use a compass and a ruler to construct a right angle in the space below.*

Definition 2 *A triangle is called right, if one of its angles is the right angle. The sides of a right triangle forming the right angle are called legs or catheti*[1]*, the side opposite to the right angle is called the hypotenuse.*

Problem 7 *Can a triangle in the Euclidean plane have more than one right angle? How about a sphere?*

[1] singular: cathetus

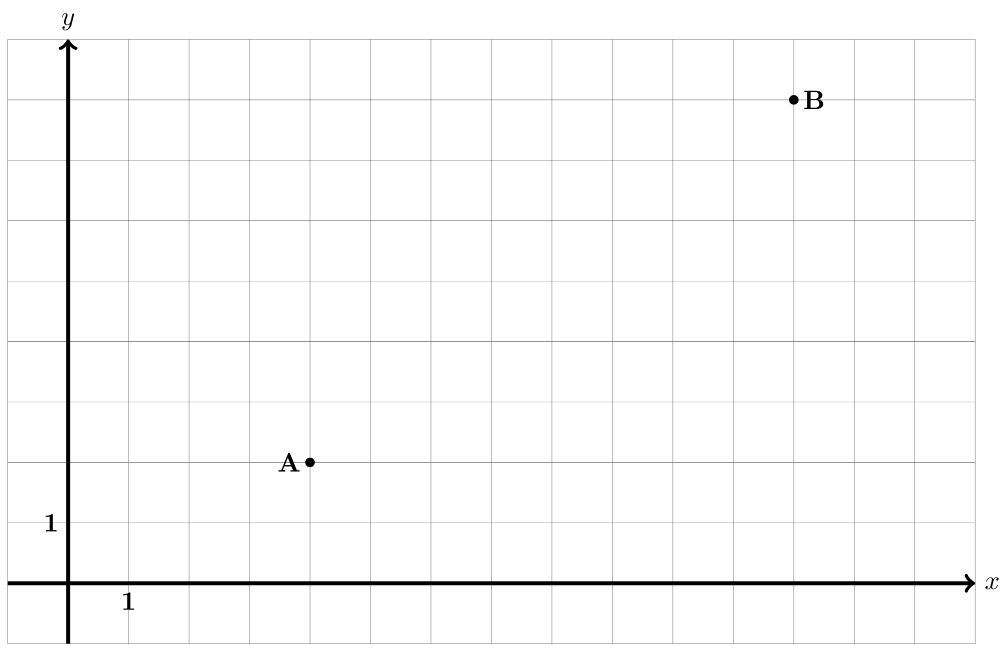
Theorem 1 (Pythagoras)
The sum of the areas of the squares built on the legs of a right triangle equals to the area of the square built on the hypotenuse.



$$a^2 + b^2 = c^2$$

Problem 8 *Prove the Pythagoras' theorem.*

Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be points in the Euclidean plane endowed with the standard coordinate system as on the picture below.



Problem 9 *Find the coordinates of the points A and B.*

$$A = (\quad , \quad) \qquad B = (\quad , \quad)$$

We will denote $|AB|$ the distance between the points A and B.

Problem 10 *Use the Pythagoras' Theorem to find $|AB|$.*

Recall that the operation

$$a^p = \underbrace{a \times a \times \ldots \times a}_{p\ times} \tag{1}$$

is called *exponentiation* or *raising to power p*. For the reason to be explained later, we will mostly consider exponentiation for the *base* $a > 0$.

Exponentiation has the following properties.

1. $a^0 = 1$.

2. $a^p \times a^q = a^{p+q}$.

3. $(a^p)^q = a^{p \times q}$.

Let $a^p = b$. The operation of *taking the pth root of b* undoes what raising to the power p (exponentiating) does.

$$\sqrt[p]{b} \stackrel{\text{def}}{\equiv} b^{\frac{1}{p}} = (a^p)^{\frac{1}{p}} = a^{p \times \frac{1}{p}} = a \tag{2}$$

For example, $\sqrt[4]{81} = 3$ because $3^4 = 81$. The root of power 2 is called the *square root*. The root of power 3 is called the *cubic root*. There are no special names for other roots. They are called the *fourth root*, *fifth root*, and so on.

Traditionally, they don't write the root power of 2 for the square root. For example, $\sqrt[2]{100} \stackrel{\text{def}}{\equiv} \sqrt{100} = 10$.

Problem 11

$2^3 =$ $\qquad 8^{\frac{1}{3}} =$ $\qquad \sqrt[3]{8} =$

$\sqrt{25} =$ $\qquad 5^2 =$ $\qquad 25^{\frac{1}{2}} =$

$\sqrt{49} =$ $\qquad 49^{\frac{1}{2}} =$ $\qquad 7^2 =$

$\sqrt[4]{16} =$ $\qquad \sqrt[5]{32} =$ $\qquad \sqrt{36} =$

$8^2 =$ $\qquad (-8)^2 =$ $\qquad (-2)^2 =$

$$\sqrt{1,000,000} =$$

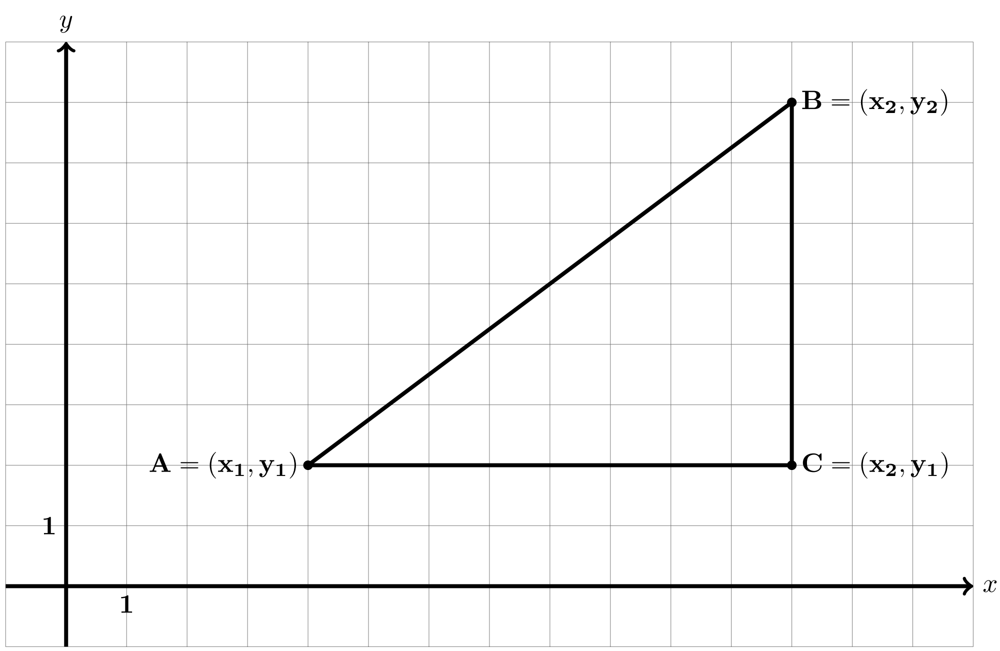
$$\sqrt[3]{1,000,000} =$$

$$\sqrt[10]{10,000,000,000} =$$

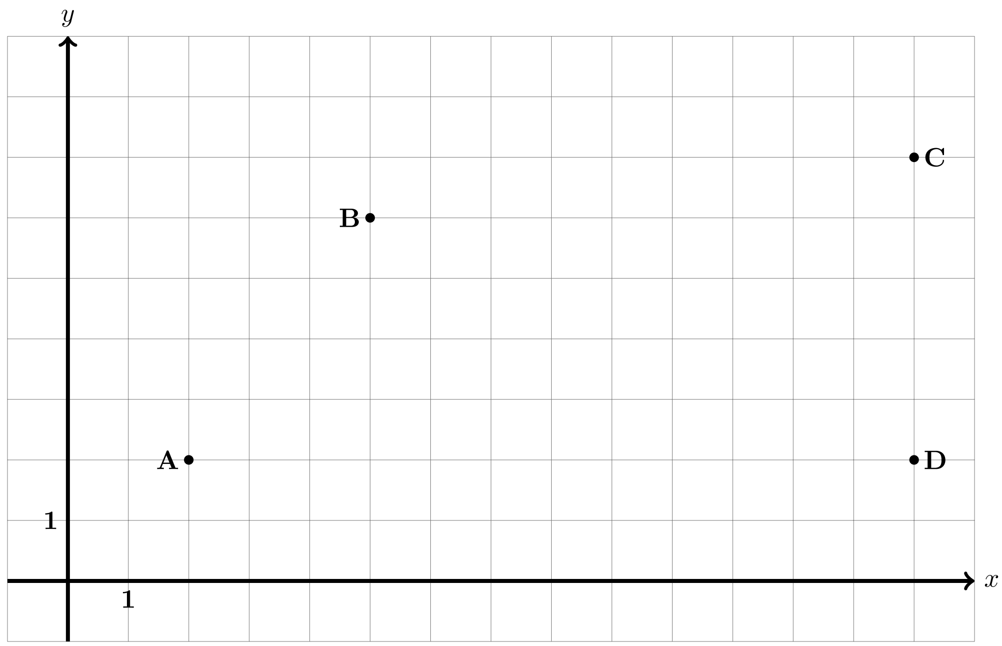
$$\sqrt[10]{1,024} =$$

To compute the distance between the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in general, let us introduce the third, auxiliary, point $C = (x_2, y_1)$. Consider the triangle ABC. The angle ACB is right, so we can employ the Pythagoras' Theorem. The area of the square built on the leg AC equals to $(x_2 - x_1)^2$. Note that the order of the coordinates does not matter, since $(x_2 - x_1)^2 = (x_1 - x_2)^2$. (Why?) Similarly, the area of the square built on the leg BC is $(y_2 - y_1)^2$. Therefore, the area of the square built on the hypotenuse AB equals to $(x_2 - x_1)^2 + (y_2 - y_1)^2$. In other words, $|AB|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. Taking the square root gives us the distance between the points A and B!

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{3}$$



Problem 12



- *Find the coordinates of the following points.*

$A = (\quad , \quad)$ $B = (\quad , \quad)$

$C = (\quad , \quad)$ $D = (\quad , \quad)$

- *Use formula 3 to find the following distances.*

$|AB| =$

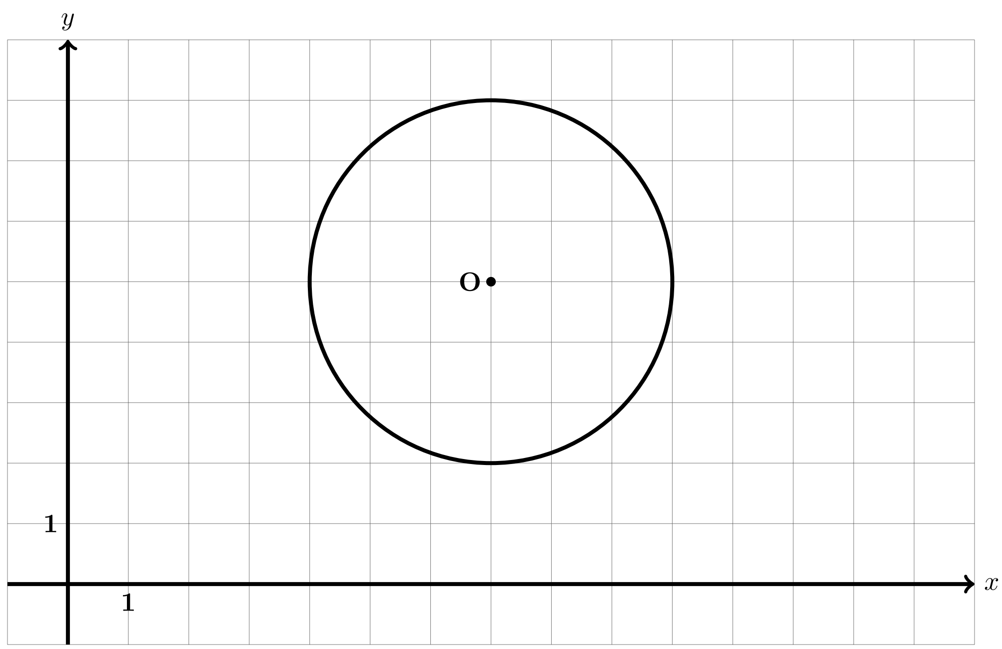
$|AC| =$

$|AD| =$

Recall that a *circumference* is the set of all the points in the plane having a specified distance, called the *radius*, from a specified point, called the *center*. Due to formula 3, the following relation holds for every point $P = (x, y)$ lying on the circumference of radius r centered at $O = (a, b)$.

$$(x - a)^2 + (y - b)^2 = r^2 \tag{4}$$

Problem 13 *Write down the equation of the following circumference.*



Does the point $Q = (4, 4.9)$ *lie on the circumference? Why or why not?*