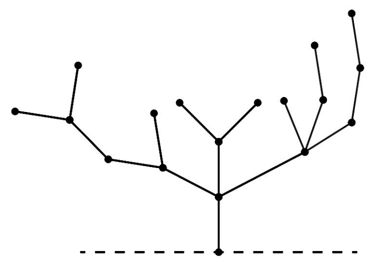
HACKENBUSH

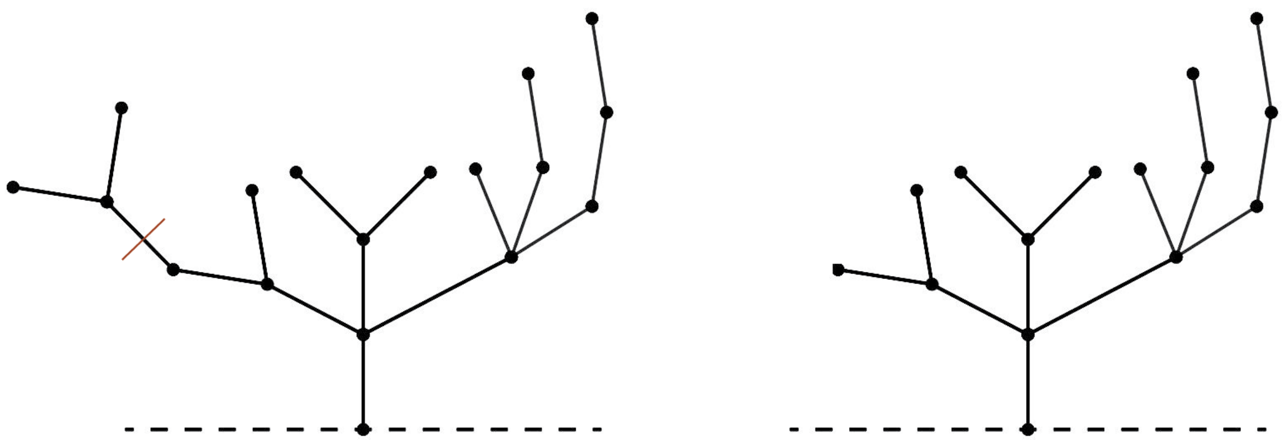
MAX STEINBERG FOR OLGA RADKO MATH CIRCLE
ADVANCED 2

1. Hackenbush

Let's play a game. Consider the following stick diagramme. On your turn, you can cross out any line (except the dashed line that marks the ground). Then, any line that is no longer connected to the ground falls off. If you have no legal moves on your turn, you lose.

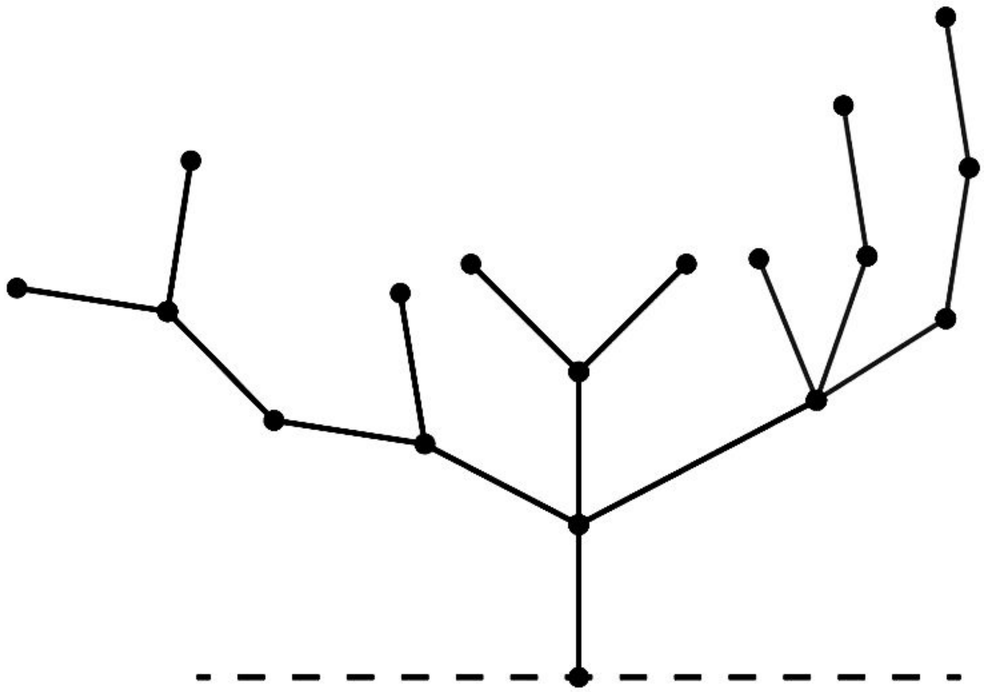


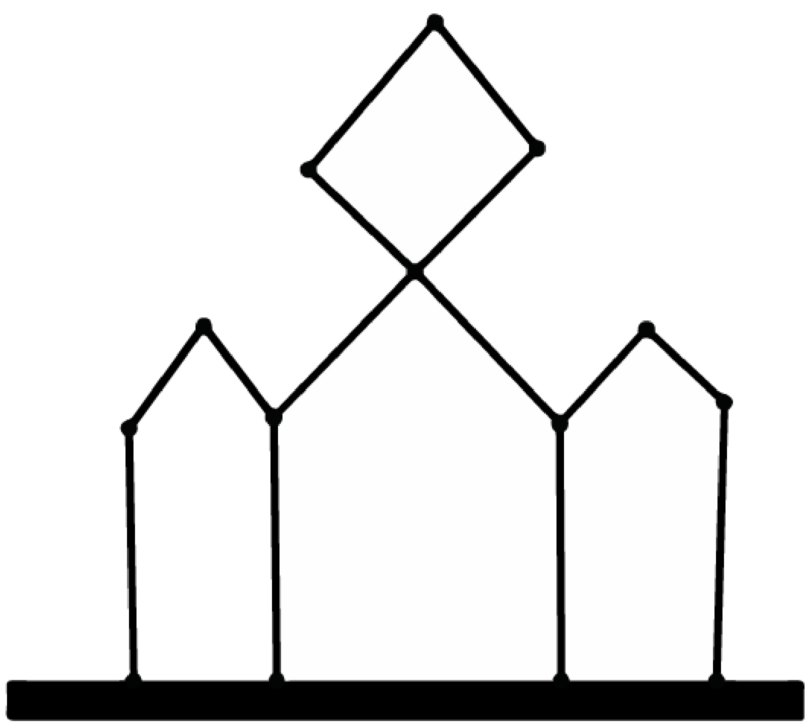
Let's look at an example move:



I made a cut shown above, and that line was removed. Then, there were two more lines not connected to the ground, so they fell off as well.

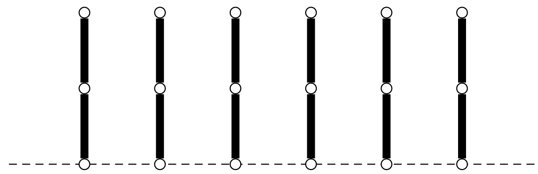
Problem 1. Play Hackenbush with the students sitting next to you. Try to find a strategy to win the game (a strategy may include selecting whether or not you wish to go first). Try playing on a variety of trees (some examples are given on the next page).





2. How to Win Hackenbush

Let's first consider a very simple example. We are playing Hackenbush and it is your turn. The current tree is a *bamboo garden* (pictured below):



Problem 2. What are all the legal moves for you to make?

Problem 3. Can you identify another game you know about with the same set of legal moves? *(Hint: think back to Week 1!)*

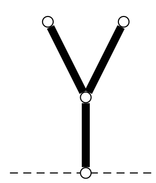
Problem 4. Who wins that game? Can you determine who would win our Hackenbush game from this position? If you don't remember how to win that game, you can check back on previous packets or ask an instructor.

Problem 5. How can you determine who wins a game of Hackenbush that starts from a bamboo garden?

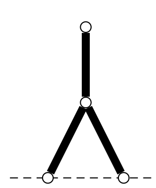
Using the ideas from the past few problems, we will assign an **evaluation** to a Hackenbush game, very similar to how we assigned a number to a Nim game. For a single bamboo stalk, the evaluation is the number of segments in the stalk. In a more complicated tree, our evaluation may be more difficult. Recall that if a player has no moves on their turn, they lose. So a bamboo stalk with 0 segments is winning for Player 2, since Player 1 loses immediately. If a bamboo stalk has k segments with $k > 0$, then Player 1 wins by cutting down the entire stalk. So our evaluation should give us an integer m, where $m = 0$ if Player 2 has a winning strategy from the current position and $m \neq 0$ if Player 1 has a winning strategy.

Problem 6. How can we assign an evaluation to a bamboo garden?

Problem 7. Consider a slightly more complicated type of tree (pictured below). How can we evaluate Hackenbush played on this tree? Why? (Feel free to play the game on the tree below, and trees similar to it, to get a feel for the strategy.)
Hint: consider a **reduction argument**: try to reduce a complicated problem to a simpler case.



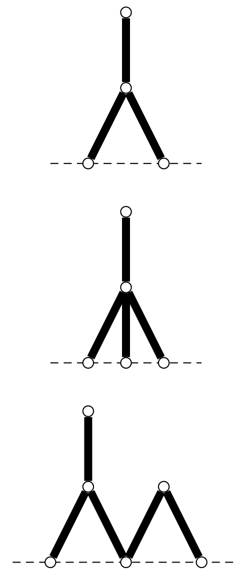
There is one additional type of complication we can have in Hackenbush trees, a **join**. A **join** is when two bamboo stalks are connected above the ground, as in the tree below.



So far, we have learned how to evaluate Hackenbush trees that are bamboo stalks or that split off into multiple branches. But what can we do to evaluate branches that join again, such as in the above tree?

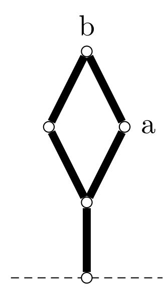
Problem 8. Who wins Hackenbush played on the above tree?

Problem 9. For each tree below, write a list of all legal moves you can make. Draw a tree without joins that has the same set of legal moves. Evaluate each of those trees.

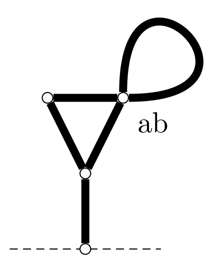


Problem 10. For each of the trees in the previous problem, determine who wins Hackenbush played on those trees. Does this align with the evaluations you obtained?

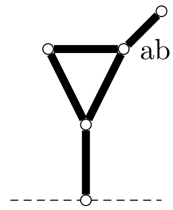
Problem 11. Please note that this problem contained a **incorrect** version of the fusion principle. For posterity, the problem is left as it was printed, although the corrected statement will also be given afterwards. The **fusion principle** states that given a loop in a Hackenbush tree, we can "fuse" two nodes together in the loop without changing the evaluation or winning strategy. Explain, in your own words, why this is true in the following example (where we fuse a and b):



$\Downarrow$



$\Downarrow$

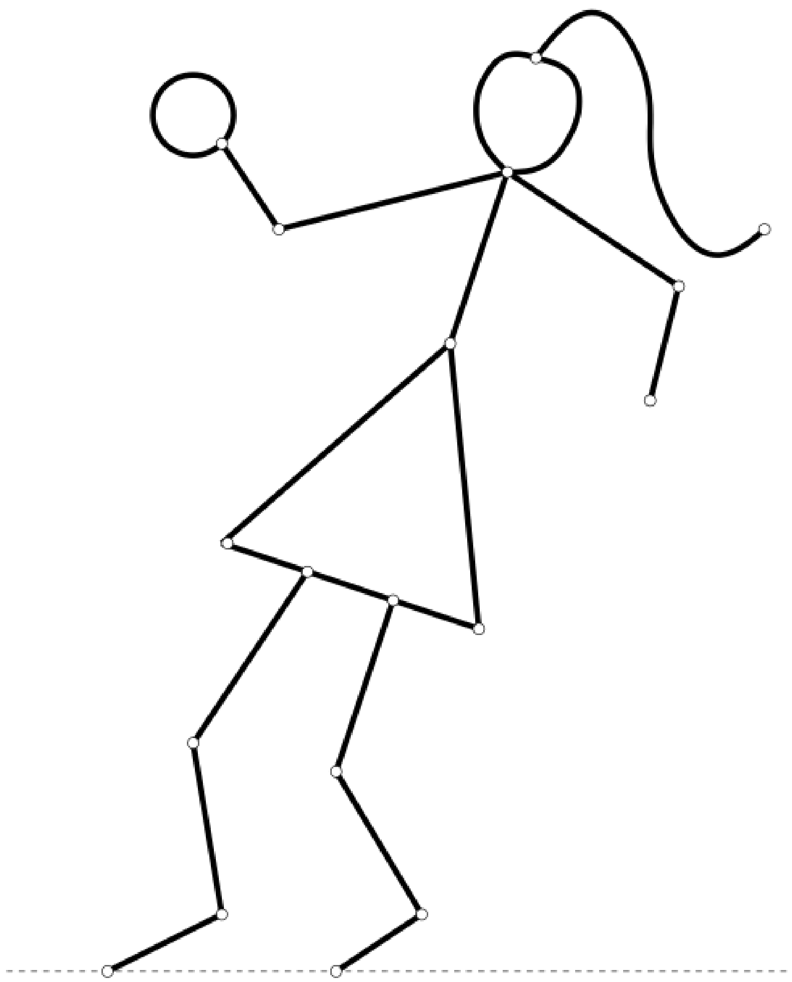


Here is the **correct** version of the statement: The **fusion principle** states that given a loop in a Hackenbush tree, we can "fuse" **all** of the nodes together in the loop without changing the evaluation or winning strategy.

The fusion principle is rather difficult to prove in general, so you can assume it to be true for the rest of the packet.

Problem 12. Evaluate the trees on Page 2.

Problem 13. Evaluate the following tree.

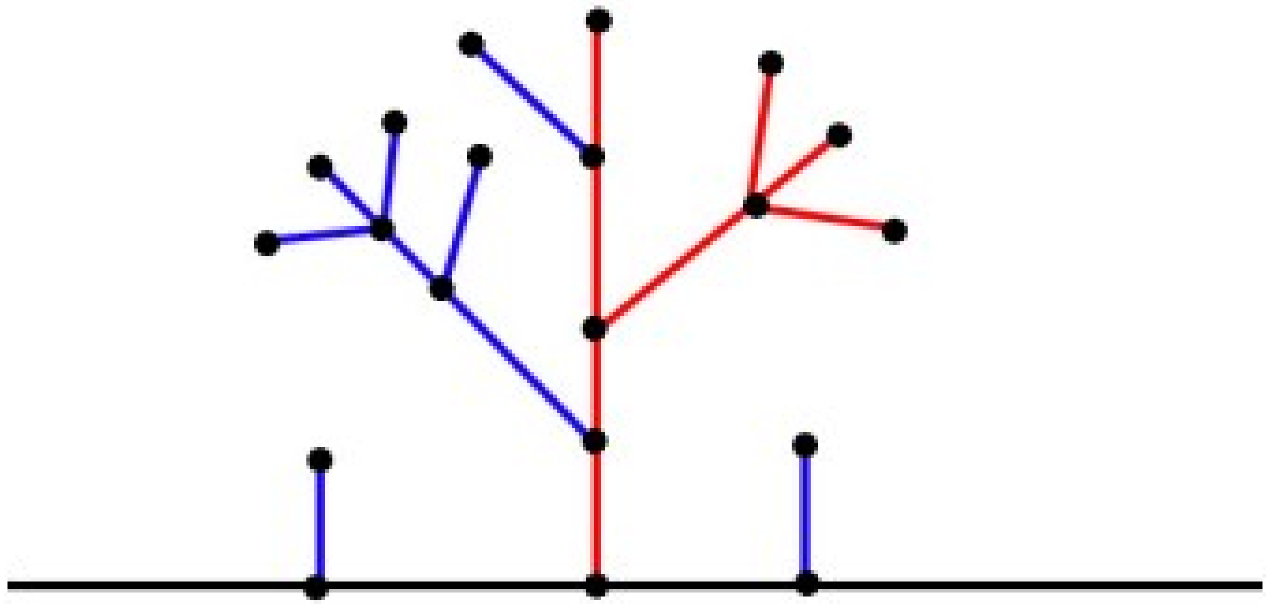


3. Difficult Problems

If you are finished doing all the above, but there still remains some time...

Problem 14. Prove the fusion principle.

Problem 15. Consider *Red-Blue Hackenbush*. One player is the Red player and one player is the Blue player. Players may only cut branches matching their colour. How can we evaluate Red-Blue Hackenbush? Try playing this game with other students who have finished the rest of the packet.



Notice that Red-Blue Hackenbush is a quite different game than normal Hackenbush. We can in fact combine the games, into *Red-Blue-Green Hackenbush*, where green branches can be cut by either player. This may appear in a future packet...