Intermediate 1 **Nov 5, 2023**

More Sets

1 Recap

Problem 1.1. *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{2, 4, 6, 8\}$*. Write down all the elements in the following sets:*

- $A \cup B$
- $A \cap B$
- $A \backslash B$
- $B \backslash A$

Problem 1.2. *Let* A *be the set of integers between 1 and 100 that are divisible by 7. What is* $|A|$*?*

Problem 1.3. *Let* $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{3, 4, 5, 6, 7, 8\}$, *and* $C = \{1, 3, 5, 7, 9\}$.

- *What is* $|A \cup B \cup C|$*?*

- *What are* $|A \cap B|, |B \cap C|$, *and* $|A \cap C|$*? What is* $|A \cap B \cap C|$*?*

- *Draw a Venn diagram and insert elements of* A, B, C *onto the corresponding part of the diagram.*

- *Confirm that the inclusion-exclusion principle holds:*

  $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Recall that the definition of the set of rationals, $\mathbb{Q}$, is

$$\mathbb{Q} = \left\{ \frac{a}{b} \ : \ a \in \mathbb{Z},\ \ b \in \mathbb{N},\ \ b \neq 0,\ \ \gcd(a,b) = 1 \right\}.$$

(What this means is that $\mathbb{Q}$ is the set of numbers $\frac{a}{b}$ such that a is an integer, b is a nonzero natural number, and the greatest common divisor of a and b is 1.)

Problem 1.4. *Write down a mathematical description of* $\mathbb{Q}\backslash\mathbb{Z}$*, similar to how we describe* $\mathbb{Q}$ *above.*

$$\mathbb{Q}\backslash\mathbb{Z} = \left\{ \frac{a}{b} \ : \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \right\}$$

Problem 1.5. *Use the notations as above, write down a proper subset of* $\mathbb{Z}$ *that has infinitely many elements.*

Problem 1.6. *What is the set* $\{x \in \mathbb{Z} \ : \ x > x + 1\}$*?*

2 Power sets

Given a set A, the *power set* of A is the set of all subsets of A. The power set of a set A is written as $\mathcal{P}(A)$.

$$\mathcal{P}(A) = \{S \,:\, S \subseteq A\}.$$

For example, if $A = \{1, 2\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Problem 2.1. *Is $\emptyset, A \in \mathcal{P}(A)$? Explain.*

Problem 2.2. *If $A = \{1, 2, 3\}$, what is $\mathcal{P}(A)$?*

Problem 2.3. *What is $\mathcal{P}(\emptyset)$?*

Problem 2.4. *Show that if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.*

Problem 2.5. *Let* $A = \{0, 1\}$*. What is* $\mathcal{P}(\mathcal{P}(A))$*?*

3 Cartesian Product

The *Cartesian product* (also called product) of two sets A and B, written as $A \times B$, is the set of all ordered pairs where the first element is in A and the second element is in B, i.e.

$$A \times B = \{(a, b) \,:\, a \in A, b \in B\}$$

We usually write $A \times A$ as A^2.

We can also have the product of many sets. $X_1 \times X_2 \times \cdots \times X_n$ is the set of ordered tuples $(x_1, x_2, \ldots, x_n)$, where the first entry is an element of X_1, the second entry is an element of X_2, and so on.

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \,:\, x_1 \in X_1, \ldots, x_n \in X_n\}.$$

Problem 3.1. *Let* $A = \{1, 2, 3\}$*,* $B = \{a, b\}$*, what is* $A \times B$*?*

Problem 3.2. *In the problem above, what is* $|A \times B|$*?*

Can you find any relationship between $|A|$, $|B|$, *and* $|A \times B|$*?*

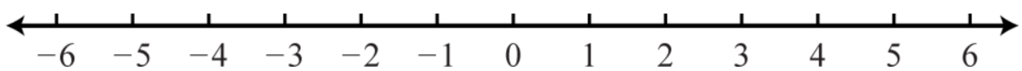
Problem 3.3. *Prove that if* $|A|$ *and* $|B|$ *are finite,* $|A \times B| = |A| \times |B|$.

Problem 3.4. *In your own words, describe the set* $\mathbb{Z} \times \mathbb{Q}$.

Visual Representation of Sets

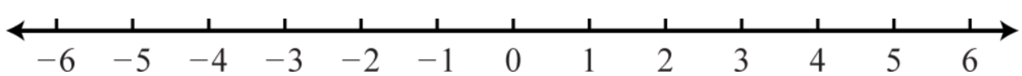
Denote the set of real numbers as $\mathbb{R}$. It contains both rationals, such as $2, \frac{1}{3}, \ldots$, and irrationals, such as $\sqrt{2}, \pi, \ldots$

A *real number line*, or simply number line, is a horizontal line where each point represents a real number, and points on the right represent bigger numbers than points on the left.



This allows us to visually display real numbers by associating them with unique points on a line. The real number associated with a point is called a *coordinate*.

Problem 3.5. *Show the points on the number line corresponding to* $-\frac{1}{2}$*,* 2*, and* π*.*

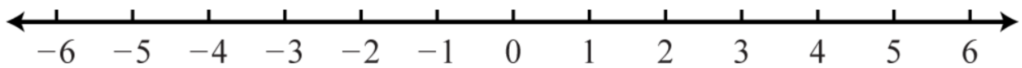


Problem 3.6. *On the same number line above, draw the subset*

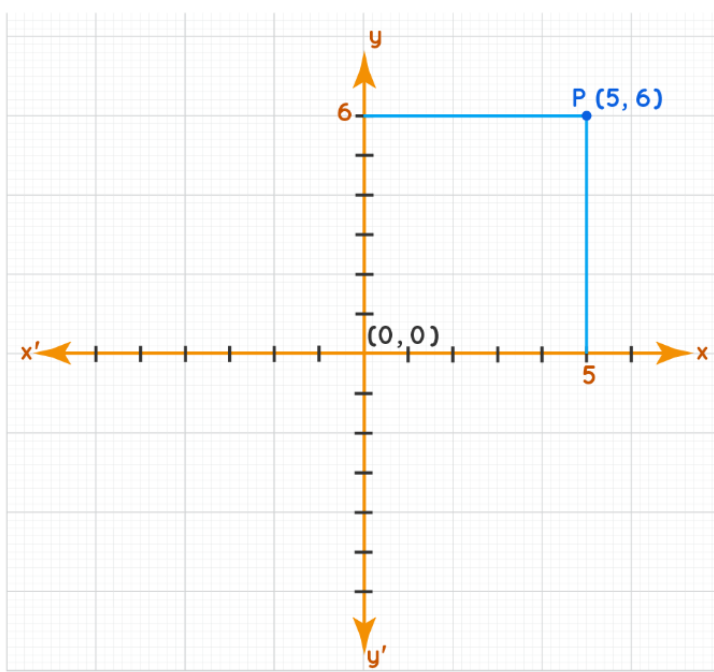
$$\{x \in \mathbb{R} \;:\; -5 \leq x \leq -2\}.$$

The set of all real numbers between a and b, including a and b, is sometimes denoted as $[a, b]$, where $a < b$. This is called a *closed interval.* For example, the set in the problem above can be denoted as $[-5, -2]$.

Problem 3.7. *A frog is jumping on the number line, starting from the point* 0. *Each jump is 1 unit long. There are food sources located in* $[-5,-3] \cup [4,6]$. *What is the minimal number of jumps the frog needs to reach food?*

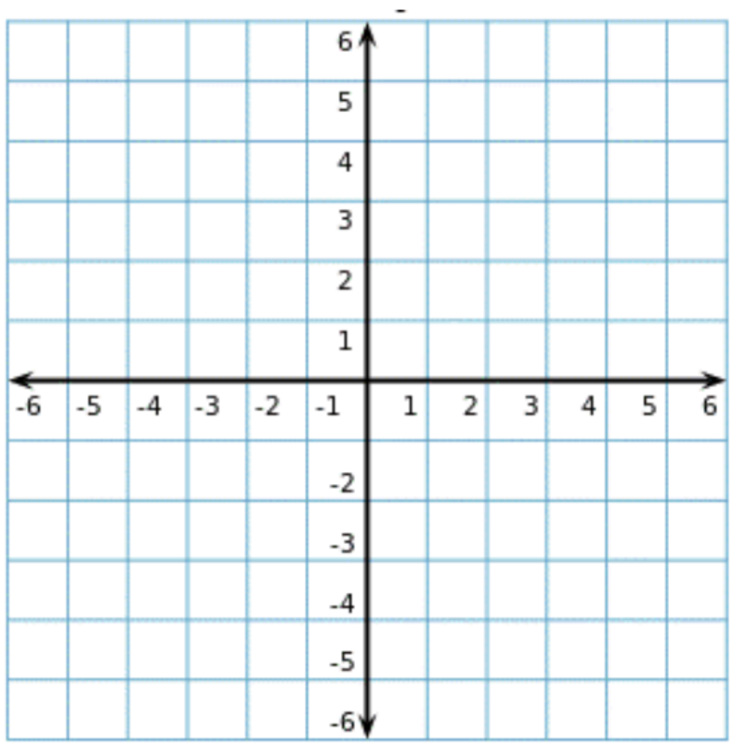


The number line is a visual representation of the set $\mathbb{R}$. A visual representation of the set $\mathbb{R}^2$ is called the *Cartesian plane*. A Cartesian plane is a plane such that each point has a unique coordinate (x, y), where x, y are real numbers.



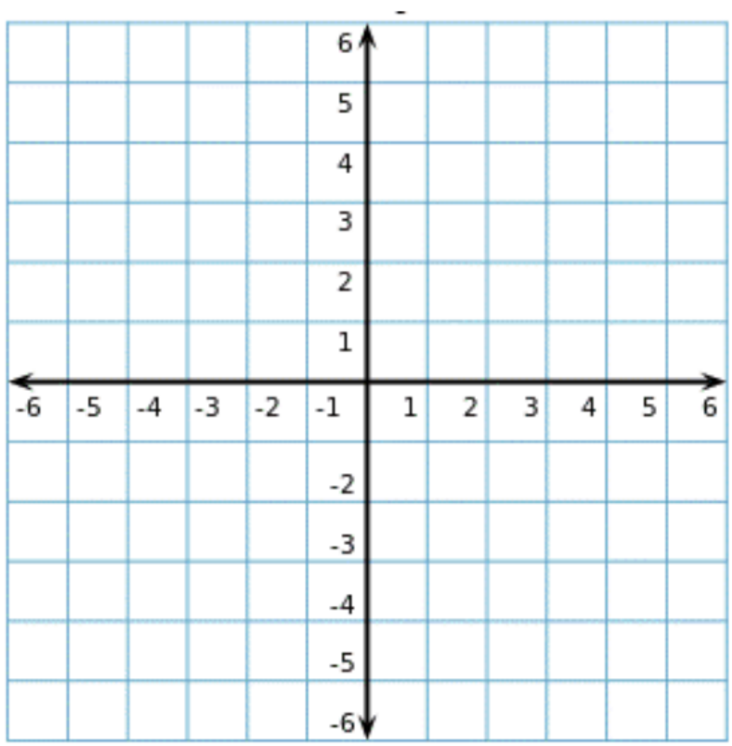
By convention, the first number x in the pair (x, y) represents position in the left/right direction (this is also called the *x-direction*), and the second number y represents the up/down direction (this is also called the *y-direction*). The point $(0, 0)$ is called the *origin* of the Cartesian plane.

Problem 3.8. *Draw the points* $(1, 0)$, $(0, 2), (-4, 5), (2, -3)$ *on the Cartesian plane below.*



Problem 3.9. *Bob walks out from his home 20 meters to the east, 40 meters to the north, then he walks 15 meters to the west, and 10 meters to the south. Use an element in* $\mathbb{R}^2$ *to represent his current position, using his home as the origin* $(0, 0)$, *east as the x-direction and north as the y-direction.*

Problem 3.10. *Indicate the region in the Cartesian plane that represents the set* $[0, 1] \times [0, 1]$*.*



Problem 3.11. *On the same plane above, show the region*

$$[2, 4] \times [3, 5] \qquad \textit{and } [3, 4.5] \times [1, 4].$$

Give a description of the intersection of the two sets using the notation $[a, b]$*.*

Problem 3.12. *Based on the visual representations of* $\mathbb{R}$ *and* $\mathbb{R}^2$ *above, what do you think* $\mathbb{R}^3$ *represents in geometry?*

Problem 3.13. *Gives a geometric object that represents the set*

$$[0,1]^3$$

Problem 3.14. *What is the geometric object represented by the following set?*

$$\{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\}$$

Problem 3.15. *Denote the set in the above problem as* $\mathbb{D}$*. What is the geometric object represented by* $\mathbb{D} \times [0,1]$*?*

Problem 3.16. *Let* S^1 *be the set of points on a circle. What is the geometric object represented by* $S^1 \times \mathbb{D}$*?*