Graph Games

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This week we're going to see a few applications of **Graph Theory** which you were introduced to last week. We'll play two games – **Sim** and **Sprouts**.

1 Sim

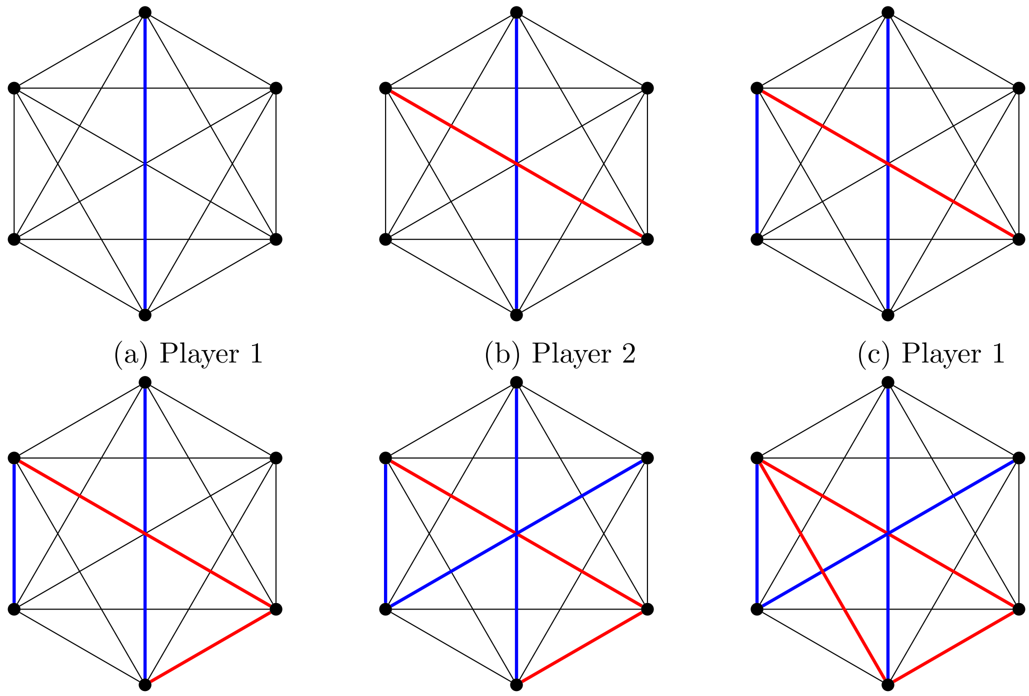
We start with Sim. Look at the hexagon shaped graph given to you along with this handout. This graph should look familiar from last week.

Problem 1.1.
What is the technical name for this graph?

1.1 Rules

The game is played as follows. Find a classmate or ask an instructor to play with you. You will need colored markers, say blue and red. The players take turns coloring the black lines. Each player should try to avoid the creation of a triangle made solely of their color. *Only triangles with the dots as all corners count; intersections of lines are not relevant.* The player who completes such a triangle **loses**.

For example, here is how a game between Player 1 and Player 2 might play out.



(a) Player 1 (b) Player 2 (c) Player 1

(d) Player 2 (e) Player 1 (f) Player 2

Player 2 loses as he creates a red triangle.

1.2 Questions

Problem 1.2.
Play a round of Sim with your partner. If you need extra sheets, ask an instructor. Is it possible for the game to end in a tie?

After playing enough games, you should be able to convince yourself that the answer to the previous problem is "no".

Problem 1.3.
Prove the following stronger statement: "Any coloring of K_6 (color some edges blue and the rest red) has a monochromatic subgraph of K_3 (a triangle whose vertices are all the same color)." Why does this imply that Sim can never end in a tie?

Problem 1.4.
Prove that if you invite 6 people to dinner, there is always a subset of 3 people who are either mutual friends or mutual strangers. (*Hint: Model this situation with a graph on 6 vertices. Each vertex represents a person and blue edges represent friendship while red edges connect strangers.*)

Problem 1.5.
Let G be an arbitrary graph with 6 vertices. Show that this graph either contains a triangle or a set of 3 vertices with no edges between them. *(Hint: Use the previous problems.)*

Definition 1.
We generalize the previous problem as follows: Define the **Ramsey number** $R(k,l)$ as the minimum number n such that any graph on n vertices contains either an independent set of size k or a clique of size l. An **independent set** is a set of vertices such that there is no edge between them. A **clique** is a set of vertices such that all pairs among them are connected by an edge.

Problem 1.6.
The previous problem showed that $R(3,3) \leq 6$. Show that it is equal to 6 i.e. if $n < 6$ then there exists a graph of n vertices such that it neither has an independent set of size 3 nor does it have a clique of size 3.

Problem 1.7 (challenge)**.**
Show that $R(m,n) = R(n,m)$ i.e. it is symmetric in its arguments. Also show that $R(1,k) = 1$ and $R(2,k) = k$.

Problem 1.8 (unsolved)**.**
Computer simulations have verified that it is always possible for the second player to win Sim. Find an efficient strategy for this.

2 Sprouts

In this section, we look at a game "Sprouts" invented by mathematicians John Conway and Michael Paterson. The setup is extremely simple but the game has numerous unsolved problems. We will prove some properties of the game using Euler's formula for planar graphs.

2.1 Rules

The game is played by two players starting with n dots drawn on a sheet of paper. For now, we will consider $n = 2$. Players take turns, where each turn consists of drawing a line between two dots (or from a dot to itself) and adding a new dot somewhere along the line. The players are constrained by the following rules:

1. The line may be straight or curved, but must not touch or cross itself or any other line.
2. The new dot cannot be placed on top of one of the endpoints of the new line. Thus the new dot splits the line into two shorter lines.
3. Each dot has exactly 3 *lives*. This means that no dot may have more than three lines attached to it. For the purposes of this rule, a line from the dot to itself counts as two attached lines and new dots are counted as having two lines already attached to them.
4. The player who makes the last move wins.

Again, let's look at an example of how a game of Sprouts might play out.

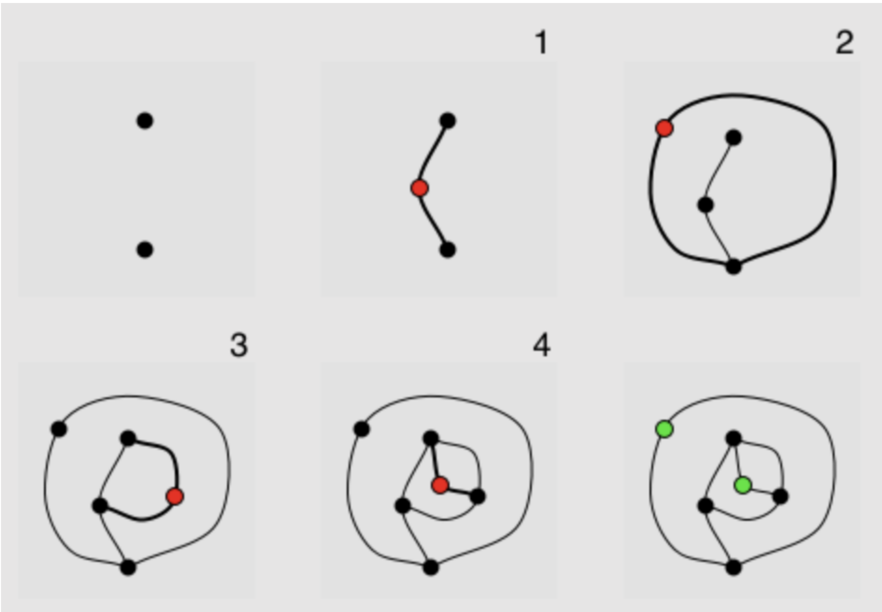


Figure 2: A 2-dot game of Sprouts

2.2 Upper bound on moves

Problem 2.1.
Grab a pen and paper and play a few rounds of Sprouts with your partner. Prove that the game always terminates for $n = 2$. Prove that it terminates for arbitrary n. (*Hint: Consider the total number of lives left for all of the dots at the end of each turn. If you can prove that this number decreases, the game must terminate.*)

Problem 2.2.
Prove that for $n = 2$ the game must terminate in at most 5 turns, for $n = 3$ the game must terminate in at most 8 turns and for arbitrary n the game must terminate in at most $3n - 1$ turns. Look at the hint for the previous problem.

Problem 2.3.
For arbitrary n, describe a game that lasts exactly $3n - 1$ moves i.e. show that the maximum from the prevous problem is achieved.

2.3 Lower bound on moves

In the next few problems, we will provide a lower bound on the number of turns that a game of Sprouts can last. This turns out to be a lot trickier than the upper bound. We will show you the proof for $n = 2$ and ask you to generalize it. We show that a game starting with 2 dots must take at least 4 turns to end.

To prove this we will use **Euler's formula for planar graphs**. Recall that last week we learned what a planar graph is. To quickly recap, a planar graph is a graph that we can draw on the page with non-overlapping edges. Note that the game of sprouts results in a planar graph[1] as we don't allow edges to cross. The formula states that for a graph G with $|E|$ edges and $|V|$ vertices, the following relation holds

$$|V| - |E| + f = 2$$

where f is the number of regions that the graph divides the plane into. For example, the game in Figure 2 gives us a graph with 4 regions. (The outer face counts too!) We guide you through the proof of this formula in Section 3. You can skip it for now and take the result for granted.

Theorem 1.
Let G be the graph obtained by playing a complete game of Sprouts starting with n dots. Let m denote the number of moves played. Then $m \geq 2n$.

Proof. We will prove this for $n = 2$. We need to show $m \geq 4$ here.

1. Let s denote the number of total *lives* (check Rule 3) left at the end of the game. Then $s = 6 - m$ as the game starts with 2 dots i.e. 6 lives and each turn reduces the number of total lives by 1 (why?).

2. Let G be the planar graph obtained after a complete game of sprouts and f the number of faces of this graph. Then $f \geq s$. This is because no region of the graph can have 2 or more free lives. If it did, the game would not be complete as a line could be drawn. Therefore, each face corresponds to at most 1 life and $f \geq s$.

3. Note that if the game starts with 2 dots and goes on for m moves then it ends with $2 + m$ dots as each move creates another dot. So, G as described above has $|V| = 2 + m$ vertices. By similar reasoning (exercise!), it has $|E| = 2m$ edges.

With the above observations, we use Euler's formula to get

$$\begin{aligned}
&|V| - |E| + f = 2\\
\Rightarrow\ &2 + m - 2m + f = 2\\
\Rightarrow\ &2 + m - 2m + s \leq 2\\
\Rightarrow\ &2 + m - 2m + 6 - m \leq 2\\
\Rightarrow\ &2m \geq 6\\
\Rightarrow\ &m \geq 3.
\end{aligned}$$

[1] Technically we get something called a multigraph which allows multiple edges between two vertices as well as edges from a vertex to itself. It turns out that this doesn't matter much as Euler's formula still holds.

We are left to show that $m \neq 3$ and then the result follows. To do this, assume for contradiction that $m = 3$. Then $f = 3$ by Euler's formula (check this) and $s = 6 - 3 = 3$ as well. Show that this leads to a contradiction. $\square$

Problem 2.4.
Prove the theorem above i.e. show that each game starting with n dots takes at least $2n$ turns to terminate. For each n describe a game that meets this minimum.

Problem 2.5 (challenge)**.**
Describe a strategy for the second player to always win when $n = 2$.

Problem 2.6 (unsolved)**.**
Mathematicians Applegate, Jacobson and Sleator worked out which player has a winning strategy up to $n = 11$ and conjectured that the first player has a winning strategy when n divided by six leaves a remainder of three, four, or five and the second player has a winning strategy in other cases. Prove or disprove their conjecture.

2.4 Brussel Sprouts

Consider a simpler version of the game named Brussels Sprouts. The game starts with n crosses, i.e. spots with four free ends (rather than dots that have 3 lives). Each move involves joining two free ends with a curve, again not crossing any existing line, and then putting a short stroke across the line to create two new free ends (as opposed to creating one free end by adding a dot in Sprouts). Note that each move removes two free ends and introduces two more.

For example, consider the following game that starts with 2 crosses.

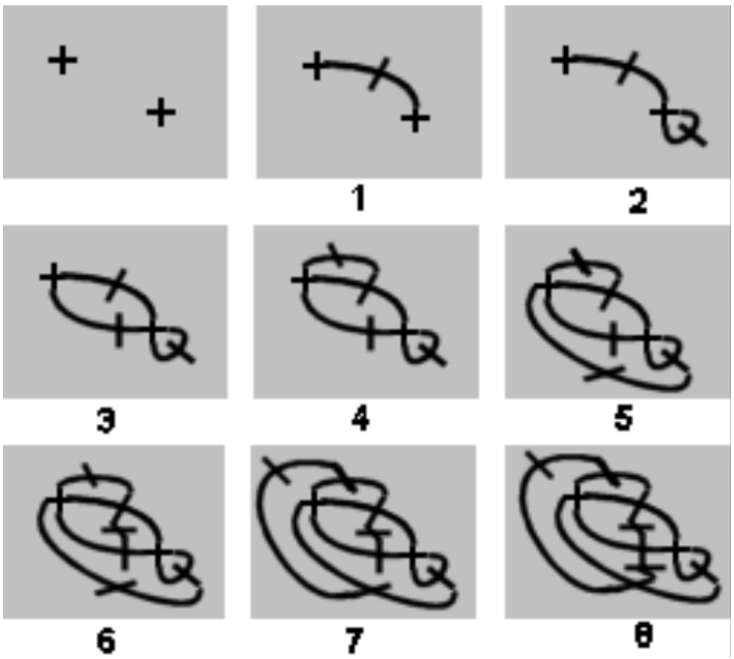


Figure 3: A 2-cross game of Brussels Sprouts. It always lasts exactly eight moves.

Problem 2.7.
Prove that with n initial crosses, the number of moves in a game will be exactly $5n - 2$. We will guide you through the proof with some hints. Again, let G denote the graph obtained after a complete game of Brussel Sprouts. Crosses and strokes are vertices while lines are edges.

1. Show that $|V| = n + m$ where m is the number of moves.
2. Show that $|E| = 2m$.
3. Show that $f = 4n$. First argue that f is the same as the number of free ends at the end of the game. Then show that this is the same as the number of free ends at the start of the game.
4. Conclude by applying Euler's formula.

Problem 2.8.
Show that this means a game starting with an odd number of crosses will be a first player win, while a game starting with an even number will be a second player win regardless of the moves.

3 Optional: Proving Euler's formula

Euler's formula relates the number of edges $|E|$, vertices $|V|$ and faces f of a planar graph as follows:

$$|V| - |E| + f = 2. \tag{1}$$

In particular, this means that for a graph drawn on paper, the number of faces is completely determined by its edges and vertices. *The number of faces does not depend on how the graph is drawn.*

For example, consider the following planar drawing of K_4 which is the complete graph on 4 vertices.

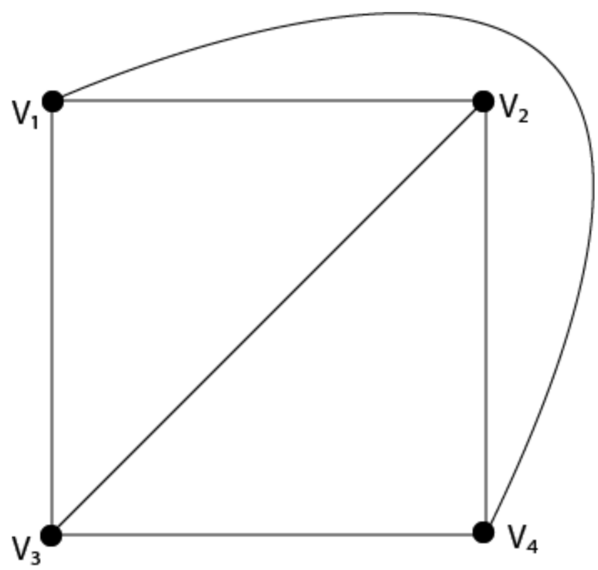


Figure 4: Planar drawing of K_4

This graph has 6 edges and divides the plane into 4 regions (recall that we count the outer region as well) so it has 4 faces. Substituting these into Euler's formula we get

$$4 - 6 + 4 = 2$$

as expected.

We will first prove a special case of this formula when the graph is a tree. A tree is defined as a (connected) graph that has no cycles (loops). Because of this, every planar drawing of a tree has exactly one face. Conversely, any connected graph with only one face has no cycles and therefore is a tree.

Here is an example of a tree.

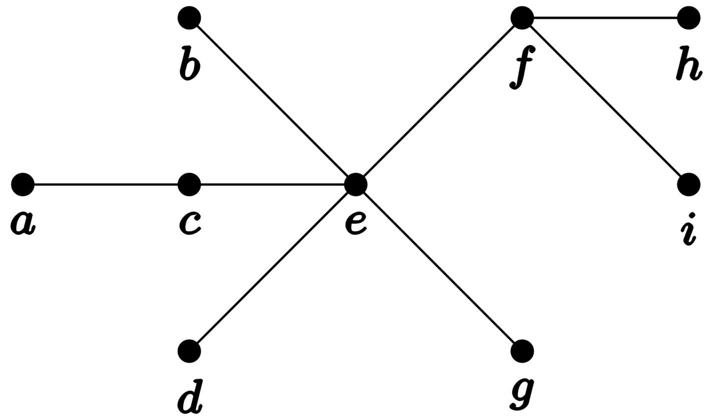


Figure 5: An example of a tree

Problem 3.1.
Draw 3 trees that are different from the one above.

Problem 3.2.
For trees, Euler's formula reduces to

$$|V| = |E| + 1.$$

Verify that this is true for the examples above.

Problem 3.3.
We call a vertex of degree 1 in a graph G a "leaf". Prove that every tree has at least two leaves. Then prove that if we remove a leaf from a tree, we are still left with a tree.

Problem 3.4.
Prove by induction that a tree with n vertices has $n-1$ edges. This proves Euler's formula for trees.

Problem 3.5.
Let $G = (V, E)$ be a connected graph with a cycle. Let e be an edge in that cycle. Prove that the graph obtained by removing e is still connected.

Problem 3.6.
Finally, prove Euler's formula for the an arbitrary connected graph (Eq 1) by induction on the number of *edges* in G. *(Hint: Distinguish the cases where G has no cycle and where it has at least one cycle. Then use your results from Problem 3.4 and Problem 3.5.)*