Sets

1 Vocabulary and notations

A *set* is a clearly defined collection of distinct objects. Note that this is not really a definition. To define means to explain in simpler terms. Instead, all we do is replacing one word, a *set*, by another, a *collection*. Plus, the meaning of the words *to clearly define* is not clearly defined. The problem is that the notion of a set is as fundamental as it is deep. It is impossible to explain it in simpler terms. The best we can do at the moment is to show a bunch of examples.

The set $F$ of all the factors of the number 6 consists of four elements, six, three, two, and one.

$$F = \{6, 3, 2, 1\}$$

Note that we use curvy braces, $\{\}$, to write down the set elements. In a set, the order of the elements does not matter,

$$F = \{1, 6, 2, 3\}.$$  

Let us list the elements of the above set in the increasing order.

$$(1, 2, 3, 6)$$

As you can see, we have switched from the braces to round brackets, $(\ast)$, a.k.a. parentheses. The round brackets are used
to write down elements of an *ordered set*, or a *list*. In a list, the order of the elements does matter. For example, let us consider two sets and two lists of letters, \( S_1 = \{C, A, T\} \), \( S_2 = \{A, C, T\} \) and \( L_1 = (C, A, T) \), \( L_2 = (A, C, T) \). Then \( S_1 = S_2 \), but \( L_1 \neq L_2 \).

The fact that the letter \( A \) is an element of the set \( S_1 \) is denoted as

\[
A \in S_1.
\]

As \( B \) is not an element of the set \( S_1 \), we write \( B \notin S_1 \).

Looking at the sets \( S_1 = \{C, A, T\} \) and \( S_3 = \{A, C\} \), one can see that every element of \( S_3 \) is also an element of \( S_1 \). In this case, \( S_3 \) is called a *subset* of \( S_1 \). In mathematical language, we write this as

\[
S_3 \subseteq S_1.
\]

**Problem 1.1.** *What is the set of colors of the United States flag?*
The set $\mathbb{N} = \{0, 1, 2, \ldots \}$ is called the set of natural numbers.

The set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$ is called the set of integral numbers, or just integers. $\mathbb{N}$ is a subset of $\mathbb{Z}$. Recall that in the math language, we write this as $\mathbb{N} \subseteq \mathbb{Z}$. Zero is an integer, so $0 \in \mathbb{Z}$.

A set without any elements is called the empty set and is denoted as $\emptyset$. The empty set is a subset of any set, including itself, $\emptyset \subseteq \emptyset$. Note that it is the empty set, not an empty set. There exists only one such.

Example 1. \{A set of pigs that can fly by themselves\} = $\emptyset$.

Problem 1.2. Give your own description of the empty set.

A subset of a set is called a proper subset if it is not equal to the original set. For example the set $S_3$ we have considered above is a proper subset of the set $S_1$. In mathematical language, we write this as $S_3 \subset S_1$. The notations $\subset$ and $\subseteq$ for sets are analogous to $<$ and $\leq$ for numbers.

Problem 1.3. True or false: $\mathbb{N} \subset \mathbb{Z}$. Explain why.
Problem 1.4. Let $S_1 = \{C, A, T\}$ and $S_2 = \{A, C, T\}$. Is $S_1$ a subset of $S_2$? Is $S_1$ a proper subset of $S_2$?

Problem 1.5. Write down all the proper subsets of the set of colors of the US flag. Do they form a set? A list?

Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. The set $\mathbb{Q}$ of all the fractions $\frac{m}{n}$ in the reduced form, a.k.a. in lowest terms, is called the set of rational numbers, denoted as $\mathbb{Q}$. Recall that the fraction $\frac{m}{n}$ is in the reduced form if $|m|$ and $n$ have no common factors except for 1. For example, $\frac{2}{3} \in \mathbb{Q}$. An integer $m$ can be considered as a fraction $\frac{m}{1}$, and thus $\mathbb{Z} \subset \mathbb{Q}$.

Problem 1.6. Is the number 0.55 rational? Why or why not?
Problem 1.7. Decide which of the following two fractions is greater without cross-multiplying or bringing to the common denominator:
\[
\frac{2022}{2023} \quad \text{or} \quad \frac{2023}{2024}
\]

Problem 1.8. True or false:

- \( \pi \in \mathbb{Q} \).
- \( 0.33333\cdots \in \mathbb{Q} \).
- \( \sqrt{2} \notin \mathbb{Q} \).

The number of elements in a set is called its \textit{cardinality}. In math language, the cardinality of a set \( A \) is denoted as either \(|A|\) or \( \text{card}(A) \). We will use the first notation in the handout. For example, \(|S_1| = 3\) for the set \( S_1 = \{C, A, T\} \), since it has exactly 3 elements.

Problem 1.9. Let \( U \) be the set of states in the United States. What is \(|U|\)?
Problem 1.10. What is $|\{0\}|$? Why?

What is $|\emptyset|$?

Problem 1.11. List all subsets, including the empty set and the entire set, of each of the following sets. How many subsets does each of them have?

$\{1\}$:

$\{1,2\}$:

$\{1,2,3\}$:

Can you find any pattern?
Problem 1.12. Given $|A| = n$, prove that the cardinality of the set of all the subsets of $A$ is $2^n$.

Problem 1.13. Given $|A| = n$, what is the cardinality of the set of all proper subsets of $A$?
2 Union and Intersection

The set of the elements that belong to the sets $A$ and $B$ is called the intersection of $A$ and $B$ and is denoted as $A \cap B$. Let us rewrite this definition completely in the math language.

$$A \cap B := \{x : x \in A \text{ and } x \in B\}. \tag{1}$$

In this mathematical sentence, ‘:=’ reads as ‘is defined as’. The colon written after the $x$ reads as ‘such that’. Translating back into English, the intersection of the sets $A$ and $B$ is defined as the set of the elements $x$ such that $x$ belongs to $A$ and $x$ belongs to $B$.

Problem 2.1. Let $A$ be the set of all the even numbers, a.k.a. the integers divisible by 2. Let $B$ be the set of all the integers divisible by 3. What is $A \cap B$?

Problem 2.2. What is $A \cap \emptyset$ for any set $A$?

The following is the definition of the union of two sets, written down in the math language.

$$A \cup B := \{x : x \in A \text{ or } x \in B\}. \tag{2}$$
Problem 2.3. Translate (2) into English.

Problem 2.4. What is $A \cup \emptyset$ for any set $A$?

Problem 2.5. Let $S_1 = \{C, A, T\}$ and $S_2 = \{A, C, T\}$. What is $S_1 \cup S_2$?

Problem 2.6. Give an example of two sets and of their union different from the ones used above.

Problem 2.7. Suppose $A$ and $B$ are two sets with $A \cup B = \emptyset$. What are $A$ and $B$?
Venn Diagram

The difference of the sets $A$ and $B$, the set $A \setminus B$, is the set of all the elements of the set $A$ that do not belong to the set $B$. In other words,

$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}.$$ 

The following picture, called a Venn diagram, helps to visualize the definition.

Problem 2.8. Show the set $A \cup B$ in the Venn diagram above.

Problem 2.9. Let $A$ be the set of spectators at a basketball game. Let $B$ be the set of all the people at the game, spectators, coaches, staff, etc., who are wearing caps. Describe in your own words the set $B \setminus A$. 
Problem 2.10. Use the symbol $\notin$ to write the definition of $B \setminus A$ in math language.

Problem 2.11. How many integers in the set

$$S = \{1, 2, 3, \ldots, 100\}$$

are not divisible by 3?

Two sets are called disjoint, if they have no elements in common. In other words, if the sets A and B are disjoint if and only if $A \cap B = \emptyset$.

Problem 2.12. Give an example of two disjoint sets.

Problem 2.13. The sets A and B are disjoint. Draw the corresponding Venn diagram.
Problem 2.14. If $A \cap B = \emptyset$, $B \cap C = \emptyset$, $A \cap C = \emptyset$, draw the corresponding Venn diagram.

Problem 2.15. Pablo asked 100 steak lovers whether they liked to put salt and pepper on their filet mignons.

According to the Venn diagram above, how many put

- Salt?
- Salt only?
- Pepper?
• Pepper only?

• Salt and pepper?

• Neither?

Problem 2.16. Gregory asked 100 kids whether they were collecting die-cast models of cars, trains, and airplanes.

According to the Venn diagram above, how many kids were collecting

• Trains?

• Planes?
• Cars?

• Trains and planes?

• Trains and planes, but not cars?

• Trains or cars, but not planes?

• All of them?

• None of them?

Example 2. There are 120 sixth grade students in a US school, each of them taking at least one world language class, in addition to English. 86 students study Spanish, 38 learn Mandarin. How many students take both classes?

Let $S$ be the set of the students taking the Spanish class. Let $M$ be the set of the students studying Mandarin. The following Venn diagram helps to visualize the problem.
We need to find the cardinality of the intersection of the two sets $|S \cap M|$. Let’s call it $x$,

$$|S \cap M| = x$$

then $|S \setminus M| = 86 - x$ and $|M \setminus S| = 38 - x$. Therefore, we have $120 = |S \setminus M| + |S \cap M| + |M \setminus S| = (86 - x) + x + (38 - x)$. This simplifies to $124 - x = 120$, so $x = 4$.

The above problem has a simpler solution. Adding up the numbers of the students taking Spanish and the students taking Mandarin, we count those who take both languages twice. Therefore, the sum $|S| + |M| = 86 + 38 = 124$ exceed the total number of the 6th grade students, 120, by $|S \cap M|$. Hence, $|S \cap M| = 4$. We just have seen the simplest case of a rather handy combinatorial technique, called the inclusion-exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$  \hspace{1cm} (3)

**Problem 2.17.** Prove (3).
Problem 2.18. In a class of 40 students, 25 of them love ice-cream and 22 of them love cheesecake. 16 of them love both ice-cream and cheesecake. How many students love neither?

Problem 2.19. Prove the following statements: (You can use the Venn diagram below as an illustration, then show it algebraically.)

1. \((A \cup B) \cap C = (A \cap C) \cup (B \cap C)\)
2. \((A \cap C) \cup B = (A \cup C) \cap (B \cup C)\)

\[
\begin{array}{c}
A \\
\cap \\
\cap \\
\cap \\
B \\
C
\end{array}
\]
Problem 2.20. Demonstrate the following inclusion-exclusion principle using a Venn diagram, then prove it algebraically:

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \]

Problem 2.21. Among 20 students in a room, 9 study Mathematics, 10 study Science, and 8 take Music classes. 4 students study Mathematics and Science, 4 take Mathematics and Music, and 3 take Science and Music. 2 students take all the three subjects. How many students take at least one of the three courses? How many take none? Solve the problem in two (slightly) different ways, using a Venn diagram and using the inclusion-exclusion principle.

Problem 2.22. How many integers between 1 and 100 are not divisible by either 2, 3, or 5?