NUMBER BASES (TAKE 10)

MATH CIRCLE (ADVANCED) 10/28/2012

Today we still study some specific number bases:

Binary (base 2) with digits $0, 1$.

Ternary (base 3) with digits $0, 1, 2$.

Octal (base 8) with digits $0, 1, 2, 3, 4, 5, 6, 7$.

Hexidecimal (base 16) with digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F$.

0) Explain the algorithm presented to convert from base 10 to base n.

Divide the number by n; the remainder is the least-significant digit. Now repeat with the quotient to get the next least-significant digit, etc.

For example,

$$21 = 10 * 2 + \mathbf{1},\ 10 = 5 * 2 + \mathbf{0},\ 5 = 2 * 2 + \mathbf{1},\ 2 = 1 * 2 + \mathbf{0},\ 1 = 0 * 2 + \mathbf{1}$$

so $21_{10} = 10101_2$. Similarly,

$$50 = 16 * 3 + \mathbf{2},\ 16 = 5 * 3 + \mathbf{1},\ 5 = 1 * 3 + \mathbf{2},\ 1 = 0 * 3 + \mathbf{1}$$

so $50_{10} = 1212_3$.

1) Fill in the following chart:

Decimal	Binary	Ternary	Octal	Hexidecimal
59	111011	2012	73	$3B$
89	1011001	10022	131	59
140	10001100	12012	214	$8C$
173	10101101	20102	255	AD

2) a) Explain the trick for converting between Binary, Octal, and Hexidecimal.

Going from right to left, every three (respectively four) digits in binary correspond to one digit in octal (respectively hexidecimal).

b) Why can't you do a similar trick for Ternary?

3 is not a power of 2.

3) Do the following conversions. Hint: You don't need to do much calculation!

a) 1025 from decimal to binary.

$1025 = 1024 + 1 = 2^{10} + 1 = 10,000,000,000_2 + 1_2 = 10,000,000,001_2$

b) 4095 from decimal to hexidecimal.

$4095 = 4096 - 1 = 1000_{16} - 1_{16} = FFF_{16}$

c) 1330 from octal to ternary.

$1330_8 = 728_{10} = 1,000,000_3 - 1_3 = 222,222_3$

4) Do the following calculations:

a) $4096_{10} \times 512_{10}$ in hexidecimal.

$1000_{16} \times 200_{16} = 200,000_{16}$

b) $129_{10} \times 31_{10}$ in binary.

$10,000,001_2 \times (100,000_2 - 1_2) = 1,000,000,100,000_2 - 10,000,001_2 = 111,110,011,111_2$

c) $242_{10} \times 80_{10}$ in ternary.

$(100,000_3 - 1_3) \times (10,000_3 - 1_3) = 1,000,000,000_3 - 100,000_3 - 10,000_3 + 1_3 = 222,120,001$

5) Prove that every natural number can be written as the difference of two numbers whose ternary representations contain only 0's and 1's.

Hint: $210_3 = 1010_3 - 100_3, 1102_3 = 1110_3 - 1_3, 2021_3 = 10101_3 - 1010_3$

6) Suppose I have four boxes with numbers (between 1 and 15) inside them.

A: $1, 3, 5, 7, 9, 11, 13, 15$

B: $2, 3, 6, 7, 10, 11, 14, 15$

C: $4, 5, 6, 7, 12, 13, 14, 15$

D: $8, 9, 10, 11, 12, 13, 14, 15$

If you pick a number between 1 and 15 and tell me all of the boxes it is in, I can tell you the number (try this if you want!). How do I do it?

Write every number between 1 and 15 as a four digit number in base 2 (adding 0's in the front as needed). A number is in box A if the last (fourth) digit is 1, box B if the third digit is 1, box C if the second digit is 1, and box D if the first digit is 1. Thus, if we associate box A with 1, B with 2, C with 4, and D with 8, the number is simply the sum of the boxes.

7) What is the minimum number of weights which enable us to weigh any integer number of grams of gold from 1 to 100 on a standard balance with two pans? Weights may be placed only on the left pan.

Hint: Think of the numbers in base 2. We claim 7 weights is enough ($1, 2, 4, 8, 16, 32, 64$). Why are 6 not sufficient?

8)* Repeat 7) if the weights can be placed on either side of the pan?

Hint: Use Problem 5 and think in base 3. We claim 5 weights is enough ($1, 3, 9, 27, 81$). Why are 4 not sufficient?

9)* Suppose $P(x)$ is an unknown polynomial, of unknown degree, with nonnegative integer coefficients. You have access to an oracle that, given an integer n, spits out $P(n)$, the value of the polynomial at n. However, the oracle charges a fee for each such computation, so you want to minimize the number of computations you ask the oracle to do. Show that it is possible to uniquely determine the polynomial after only two consultations of the oracle.

Hint: First ask for $P(1)$. Then choose $b > P(1)$ and ask for $P(b)$. This will allow you to ge the coefficients of the polynomial. Why? Think in base b.

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"