Sets

1 Vocabulary and notations

A set is a clearly defined collection of distinct objects. Note that this is not really a definition. To define means to explain in simpler terms. Instead, all we do is replacing one word, a set, by another, a collection. Plus, the meaning of the words to clearly define is not clearly defined. The problem is that the notion of a set is as fundamental as it is deep. It is impossible to explain it in simpler terms. The best we can do at the moment is to show a bunch of examples.

The set $F$ of all the factors of the number 6 consists of four elements, six, three, two, and one.

$$F = \{6, 3, 2, 1\}$$

Note that we use curvy braces, $\{\}$, to write down the set elements. In a set, the order of the elements does not matter,

$$F = \{1, 6, 2, 3\}.$$

Let us list the elements of the above set in the increasing order.

$$(1, 2, 3, 6)$$

As you can see, we have switched from the braces to round brackets, $()$, a.k.a. parentheses. The round brackets are used
to write down elements of an ordered set, or a list. In a list, the order of the elements does matter. For example, let us consider two sets and two lists of letters, $S_1 = \{C, A, T\}$, $S_2 = \{A, C, T\}$ and $L_1 = (C, A, T)$, $L_2 = (A, C, T)$. Then $S_1 = S_2$, but $L_1 \neq L_2$.

The fact that the letter $A$ is an element of the set $S_1$ is denoted as

$$A \in S_1.$$  

As $B$ is not an element of the set $S_1$, we write $B \notin S_1$.

Looking at the sets $S_1 = \{C, A, T\}$ and $S_3 = \{A, C\}$, one can see that every element of $S_3$ is also an element of $S_1$. In this case, $S_3$ is called a subset of $S_1$. In mathematical language, we write this as

$$S_3 \subseteq S_1.$$  

**Problem 1.1.** *What is the set of colors of the United States flag?*
The set \( \mathbb{N} = \{0, 1, 2, \ldots \} \) is called the set of natural numbers.

The set \( \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \) is called the set of integral numbers, or just integers. \( \mathbb{N} \) is a subset of \( \mathbb{Z} \). Recall that in the math language, we write this as \( \mathbb{N} \subseteq \mathbb{Z} \). Zero is an integer, so \( 0 \in \mathbb{Z} \).

A set without any elements is called the empty set and is denoted as \( \emptyset \). The empty set is a subset of any set, including itself, \( \emptyset \subseteq \emptyset \). Note that it is the empty set, not an empty set. There exists only one such.

**Example 1.** \( \{ \text{A set of pigs that can fly by themselves} \} = \emptyset \).

**Problem 1.2.** Give your own description of the empty set.

A subset of a set is called a *proper subset* if it is not equal to the original set. For example the set \( S_3 \) we have considered above is a proper subset of the set \( S_1 \). In mathematical language, we write this as \( S_3 \subset S_1 \). The notations \( \subset \) and \( \subseteq \) for sets are analogous to \( < \) and \( \leq \) for numbers.

**Problem 1.3.** True or false: \( \mathbb{N} \subset \mathbb{Z} \). Explain why.
Problem 1.4. Let \( S_1 = \{C, A, T\} \) and \( S_2 = \{A, C, T\} \). Is \( S_1 \) a subset of \( S_2 \)? Is \( S_1 \) a proper subset of \( S_2 \)?

Problem 1.5. Write down all the proper subsets of the set of colors of the US flag. Do they form a set? A list?

Let \( m \in \mathbb{Z} \) and \( n \in \mathbb{N}, n \neq 0 \). The set \( \mathbb{Q} \) of all the fractions \( \frac{m}{n} \) in the reduced form, a.k.a. in lowest terms, is called the set of rational numbers, denoted as \( \mathbb{Q} \). Recall that the fraction \( \frac{m}{n} \) is in the reduced form if \(|m|\) and \(n\) have no common factors except for 1. For example, \( \frac{2}{3} \in \mathbb{Q} \). An integer \( m \) can be considered as a fraction \( \frac{m}{1} \), and thus \( \mathbb{Z} \subset \mathbb{Q} \).

Problem 1.6. Is the number 0.55 rational? Why or why not?
Problem 1.7. Decide which of the following two fractions is greater without cross-multiplying or bringing to the common denominator:

\[
\frac{2022}{2023} \quad \text{or} \quad \frac{2023}{2024}
\]

Problem 1.8. True or false:

- \(\pi \in \mathbb{Q}\).
- \(0.3333\ldots \in \mathbb{Q}\).
- \(\sqrt{2} \notin \mathbb{Q}\).

The number of elements in a set is called its cardinality. In math language, the cardinality of a set \(A\) is denoted as either \(|A|\) or \(\text{card}(A)\). We will use the first notation in the handout. For example, \(|S_1| = 3\) for the set \(S_1 = \{C, A, T\}\), since it has exactly 3 elements.

Problem 1.9. Let \(U\) be the set of states in the United States. What is \(|U|\)?
Problem 1.10. **What is |\{0\}|? Why?**

\[\text{What is } |\emptyset|?\]

Problem 1.11. **List all subsets, including the empty set and the entire set, of each of the following sets. How many subsets does each of them have?**

\[
\begin{align*}
\{1\} : & \\
\{1, 2\} : & \\
\{1, 2, 3\} : & \\
\end{align*}
\]

*Can you find any pattern?*
Problem 1.12. Given $|A| = n$, prove that the cardinality of the set of all the subsets of $A$ is $2^n$.

Problem 1.13. Given $|A| = n$, what is the cardinality of the set of all proper subsets of $A$?
2 Union and Intersection

The set of the elements that belong to the sets $A$ and $B$ is called the intersection of $A$ and $B$ and is denoted as $A \cap B$. Let us rewrite this definition completely in the math language.

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$  \hfill (1)

In this mathematical sentence, ‘:=’ reads as ‘is defined as’. The colon written after the $x$ reads as ‘such that’. Translating back into English, the intersection of the sets $A$ and $B$ is defined as the set of the elements $x$ such that $x$ belongs to $A$ and $x$ belongs to $B$.

**Problem 2.1.** Let $A$ be the set of all the even numbers, a.k.a. the integers divisible by 2. Let $B$ be the set of all the integers divisible by 3. What is $A \cap B$?

**Problem 2.2.** What is $A \cap \emptyset$ for any set $A$?

The following is the definition of the union of two sets, written down in the math language.

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$  \hfill (2)
Problem 2.3. Translate (2) into English.

Problem 2.4. What is $A \cup \emptyset$ for any set $A$?

Problem 2.5. Let $S_1 = \{C, A, T\}$ and $S_2 = \{A, C, T\}$. What is $S_1 \cup S_2$?

Problem 2.6. Give an example of two sets and of their union different from the ones used above.

Problem 2.7. Suppose $A$ and $B$ are two sets with $A \cup B = \emptyset$. What are $A$ and $B$?
**Venn Diagram**

The difference of the sets $A$ and $B$, the set $A\setminus B$, is the set of all the elements of the set $A$ that do not belong to the set $B$. In other words,

$$A\setminus B := \{ x : x \in A \text{ and } x \notin B \}.$$  

The following picture, called a Venn diagram, helps to visualize the definition.

\begin{center}
\begin{tikzpicture}
  \node at (1,0) (A) {$A \setminus B$};
  \node at (0,0) (B) {$A \cap B$};
  \node at (2,0) (C) {$B \setminus A$};
  \draw (A) -- (B) -- (C) -- cycle;
\end{tikzpicture}
\end{center}

**Problem 2.8.** Show the set $A \cup B$ in the Venn diagram above.

**Problem 2.9.** Let $A$ be the set of spectators at a basketball game. Let $B$ be the set of all the people at the game, spectators, coaches, staff, etc., who are wearing caps. Describe in your own words the set $B \setminus A$. 
Problem 2.10. Use the symbol $\notin$ to write the definition of $B \setminus A$ in math language.

Problem 2.11. How many integers in the set
\[ S = \{1, 2, 3, \ldots, 100\} \]
are not divisible by 3?

Two sets are called disjoint, if they have no elements in common. In other words, if the sets $A$ and $B$ are disjoint if and only if $A \cap B = \emptyset$.

Problem 2.12. Give an example of two disjoint sets.

Problem 2.13. The sets $A$ and $B$ are disjoint. Draw the corresponding Venn diagram.
Problem 2.14. If \( A \cap B = \emptyset, \ B \cap C = \emptyset, \ A \cap C = \emptyset \), draw the corresponding Venn diagram.

Problem 2.15. Pablo asked 100 steak lovers whether they liked to put salt and pepper on their filet mignons.

![Venn Diagram]

According to the Venn diagram above, how many put

- Salt?

- Salt only?

- Pepper?
• Pepper only?

• Salt and peper?

• Neither?

**Problem 2.16.** Gregory asked 100 kids whether they were collecting die-cast models of cars, trains, and airplanes.

According to the Venn diagram above, how many kids were collecting

• Trains?

• Planes?
Example 2. There are 120 sixth grade students in a US school, each of them taking at least one world language class, in addition to English. 86 students study Spanish, 38 learn Mandarin. How many students take both classes?

Let $S$ be the set of the students taking the Spanish class. Let $M$ be the set of the students studying Mandarin. The following Venn diagram helps to visualize the problem.

\[
S \setminus M \quad S \cap M \quad M \setminus S
\]
We need to find the cardinality of the intersection of the two sets $|S \cap M|$. Let’s call it $x$,

$$|S \cap M| = x$$

then $|S \setminus M| = 86 - x$ and $|M \setminus S| = 38 - x$. Therefore, we have $120 = |S \setminus M| + |S \cap M| + |M \setminus S| = (86 - x) + x + (38 - x)$. This simplifies to $124 - x = 120$, so $x = 4$.

The above problem has a simpler solution. Adding up the numbers of the students taking Spanish and the students taking Mandarin, we count those who take both languages twice. Therefore, the sum $|S| + |M| = 86 + 38 = 124$ exceed the total number of the 6th grade students, 120, by $|S \cap M|$. Hence, $|S \cap M| = 4$. We just have seen the simplest case of a rather handy combinatorial technique, called the inclusion-exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(3)

**Problem 2.17.** Prove (3).
Problem 2.18. In a class of 40 students, 25 of them love ice-cream and 22 of them love cheesecake. 16 of them love both ice-cream and cheesecake. How many students love neither?

Problem 2.19. Prove the following statements: (You can use the Venn diagram below as an illustration, then show it algebraically.)

1. \((A \cup B) \cap C = (A \cap C) \cup (B \cap C)\)
2. \((A \cap C) \cup B = (A \cup B) \cap (B \cup C)\)
Problem 2.20. Demonstrate the following inclusion-exclusion principle using a Venn diagram, then prove it algebraically:

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \]

Problem 2.21. Among 20 students in a room, 9 study Mathematics, 10 study Science, and 8 take Music classes. 4 students study Mathematics and Science, 4 take Mathematics and Music, and 3 take Science and Music. 2 students take all the three subjects. How many students take at least one of the three courses? How many take none? Solve the problem in two (slightly) different ways, using a Venn diagram and using the inclusion-exclusion principle.

Problem 2.22. How many integers between 1 and 100 are not divisible by either 2, 3, or 5?