Intermediate 1 **Nov 5, 2023**

Sets

1 Vocabulary and notations

A *set* is a clearly defined collection of distinct objects. Note that this is not really a definition. To define means to explain in simpler terms. Instead, all we do is replacing one word, a *set*, by another, a *collection*. Plus, the meaning of the words *to clearly define* is not clearly defined. The problem is that the notion of a set is as fundamental as it is deep. It is impossible to explain it in simpler terms. The best we can do at the moment is to show a bunch of examples.

The set F of all the factors of the number 6 consists of four elements, six, three, two, and one.

$$F = \{6, 3, 2, 1\}$$

Note that we use curvy braces, $\{*\}$, to write down the set elements. In a set, the order of the elements does not matter,

$$F = \{1, 6, 2, 3\}.$$

Let us list the elements of the above set in the increasing order.

$$(1, 2, 3, 6)$$

As you can see, we have switched from the braces to round brackets, $(*)$, a.k.a. parentheses. The round brackets are used to write down elements of an *ordered set*, or a *list*. In a list, the order of the elements does matter. For example, let us consider two sets and two lists of letters, $S_1 = \{C, A, T\}$, $S_2 = \{A, C, T\}$ and $L_1 = (C, A, T)$, $L_2 = (A, C, T)$. Then $S_1 = S_2$, but $L_1 \neq L_2$.

The fact that the letter A is an element of the set S_1 is denoted as

$$A \in S_1.$$

As B is not an element of the set S_1, we write $B \notin S_1$.

Looking at the sets $S_1 = \{C, A, T\}$ and $S_3 = \{A, C\}$, one can see that every element of S_3 is also an element of S_1. In this case, S_3 is called a *subset* of S_1. In mathematical language, we write this as

$$S_3 \subseteq S_1.$$

Problem 1.1. *What is the set of colors of the United States flag?*

The set $\mathbb{N} = \{0, 1, 2, \dots\}$ is called the set of natural numbers.

The set $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is called the set of integral numbers, or just integers. $\mathbb{N}$ is a subset of $\mathbb{Z}$. Recall that in the math language, we write this as $\mathbb{N} \subseteq \mathbb{Z}$. Zero is an integer, so $0 \in \mathbb{Z}$.

A set without any elements is called the empty set and is denoted as $\emptyset$. The empty set is a subset of any set, including itself, $\emptyset \subseteq \emptyset$. Note that it is **the** empty set, not **an** empty set. There exists only one such.

Example 1. $\{$*A set of pigs that can fly by themselves*$\} = \emptyset$.

Problem 1.2. *Give your own description of the empty set.*

A subset of a set is called a *proper subset* if it is not equal to the original set. For example the set S_3 we have considered above is a proper subset of the set S_1. In mathematical language, we write this as $S_3 \subset S_1$. The notations $\subset$ and $\subseteq$ for sets are analogous to $<$ and $\leq$ for numbers.

Problem 1.3. *True or false:* $\mathbb{N} \subset \mathbb{Z}$. *Explain why.*

Problem 1.4. *Let $S_1 = \{C, A, T\}$ and $S_2 = \{A, C, T\}$. Is S_1 a subset of S_2? Is S_1 a proper subset of S_2?*

Problem 1.5. *Write down all the proper subsets of the set of colors of the US flag. Do they form a set? A list?*

Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, $n \neq 0$. The set $\mathbb{Q}$ of all the fractions $\frac{m}{n}$ in the reduced form, a.k.a. in lowest terms, is called the set of rational numbers, denoted as $\mathbb{Q}$. Recall that the fraction $\frac{m}{n}$ is in the reduced form if $|m|$ and n have no common factors except for 1. For example, $\frac{2}{3} \in \mathbb{Q}$. An integer m can be considered as a fraction $\frac{m}{1}$, and thus $\mathbb{Z} \subset \mathbb{Q}$.

Problem 1.6. *Is the number 0.55 rational? Why or why not?*

Problem 1.7. *Decide which of the following two fractions is greater without cross-multiplying or bringing to the common denominator:*

$$\frac{2022}{2023} \qquad or \qquad \frac{2023}{2024}$$

Problem 1.8. *True or false:*

- $\pi \in \mathbb{Q}$.
- $0.33333\cdots \in \mathbb{Q}$.
- $\sqrt{2} \notin \mathbb{Q}$.

The number of elements in a set is called its *cardinality*. In math language, the cardinality of a set A is denoted as either $|A|$ or $\mathrm{card}(A)$. We will use the first notation in the handout. For example, $|S_1| = 3$ for the set $S_1 = \{C, A, T\}$, since it has exactly 3 elements.

Problem 1.9. *Let U be the set of states in the United States. What is $|U|$?*

Problem 1.10. *What is* $|\{0\}|$*? Why?*

What is $|\emptyset|$*?*

Problem 1.11. *List all subsets, including the empty set and the entire set, of each of the following sets. How many subsets does each of them have?*

$$\{1\}:$$

$$\{1,2\}:$$

$$\{1,2,3\}:$$

Can you find any pattern?

Problem 1.12. *Given $|A| = n$, prove that the cardinality of the set of all the subsets of A is 2^n.*

Problem 1.13. *Given $|A| = n$, what is the cardinality of the set of all proper subsets of A?*

2 Union and Intersection

The set of the elements that belong to the sets A and B is called the *intersection* of A and B and is denoted as $A \cap B$. Let us rewrite this definition completely in the math language.

$$A \cap B := \{x : x \in A \text{ and } x \in B\}. \tag{1}$$

In this mathematical sentence, ':=' reads as '*is defined as*'. The colon written after the x reads as '*such that*'. Translating back into English, the *intersection of the sets A and B is defined as the set of the elements x such that x belongs to A and x belongs to B.*

Problem 2.1. *Let A be the set of all the even numbers, a.k.a. the integers divisible by 2. Let B be the set of all the integers divisible by 3. What is $A \cap B$?*

Problem 2.2. *What is $A \cap \emptyset$ for any set A?*

The following is the definition of the *union* of two sets, written down in the math language.

$$A \cup B := \{x : x \in A \text{ or } x \in B\}. \tag{2}$$

Problem 2.3. *Translate (2) into English.*

Problem 2.4. *What is* $A \cup \emptyset$ *for any set* A*?*

Problem 2.5. *Let* $S_1 = \{C, A, T\}$ *and* $S_2 = \{A, C, T\}$*. What is* $S_1 \cup S_2$*?*

Problem 2.6. *Give an example of two sets and of their union different from the ones used above.*

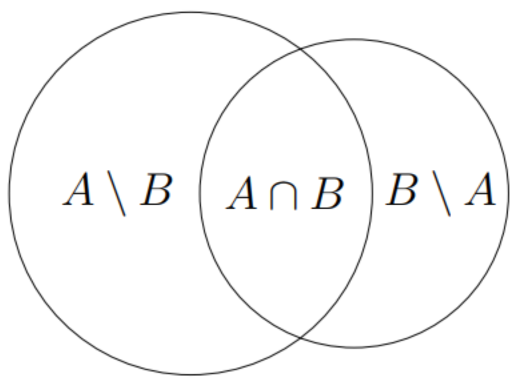
Problem 2.7. *Suppose* A *and* B *are two sets with* $A \cup B = \emptyset$*. What are* A *and* B*?*

Venn Diagram

The difference of the sets A and B, the set $A \backslash B$, is the set of all the elements of the set A that do not belong to the set B. In other words,

$$A \backslash B := \{x \,:\, x \in A \text{ and } x \notin B\}.$$

The following picture, called a Venn diagram, helps to visualize the definition.



Problem 2.8. *Show the set $A \cup B$ in the Venn diagram above.*

Problem 2.9. *Let A be the set of spectators at a basketball game. Let B be the set of all the people at the game, spectators, coaches, staff, etc., who are wearing caps. Describe in your own words the set $B \backslash A$.*

Problem 2.10. *Use the symbol* $\notin$ *to write the definition of* $B \backslash A$ *in math language.*

Problem 2.11. *How many integers in the set*

$$S = \{1, 2, 3, \dots, 100\}$$

are not divisible by 3?

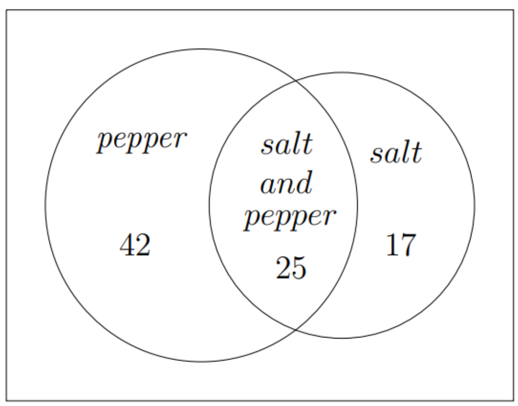
Two sets are called *disjoint*, if they have no elements in common. In other words, if the sets A and B are disjoint if and only if $A \cap B = \emptyset$.

Problem 2.12. *Give an example of two disjoint sets.*

Problem 2.13. *The sets A and B are disjoint. Draw the corresponding Venn diagram.*

Problem 2.14. *If $A \cap B = \emptyset$, $B \cap C = \emptyset$, $A \cap C = \emptyset$, draw the corresponding Venn diagram.*

Problem 2.15. *Pablo asked 100 steak lovers whether they liked to put salt and pepper on their filet mignons.*



According to the Venn diagram above, how many put

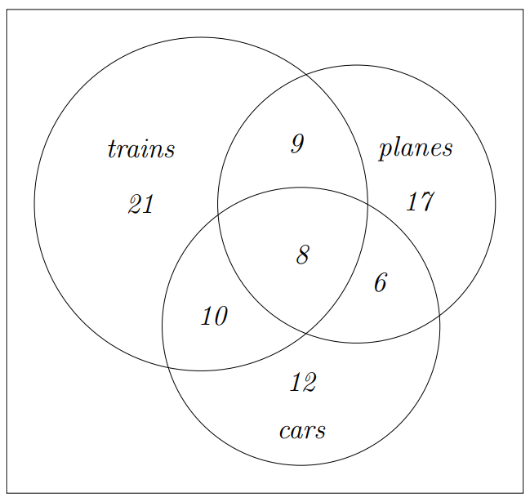
- *Salt?*
- *Salt only?*
- *Pepper?*

- *Pepper only?*

- *Salt and peper?*

- *Neither?*

Problem 2.16. *Gregory asked 100 kids whether they were collecting die-cast models of cars, trains, and airplanes.*



According to the Venn diagram above, how many kids were collecting

- *Trains?*

- *Planes?*

- *Cars?*

- *Trains and planes?*

- *Trains and planes, but not cars?*

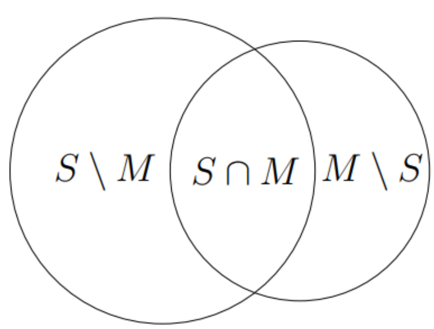
- *Trains **or** cars, but not planes?*

- *All of them?*

- *None of them?*

Example 2. *There are 120 sixth grade students in a US school, each of them taking at least one world language class, in addition to English. 86 students study Spanish, 38 learn Mandarin. How many students take both classes?*

Let S be the set of the students taking the Spanish class. Let M be the set of the students studying Mandarin. The following Venn diagram helps to visualize the problem.



We need to find the cardinality of the intersection of the two sets $|S \cap M|$. Let's call it x,

$$|S \cap M| = x$$

then $|S \backslash M| = 86 - x$ and $|M \backslash S| = 38 - x$. Therefore, we have $120 = |S \backslash M| + |S \cap M| + |M \backslash S| = (86 - x) + x + (38 - x)$. This simplifies to $124 - x = 120$, so $x = 4$.

The above problem has a simpler solution. Adding up the numbers of the students taking Spanish and the students taking Mandarin, we count those who take both languages twice. Therefore, the sum $|S| + |M| = 86 + 38 = 124$ exceed the total number of the 6th grade students, 120, by $|S \cap M|$. Hence, $|S \cap M| = 4$. We just have seen the simplest case of a rather handy combinatorial technique, called the *inclusion-exclusion principle*:

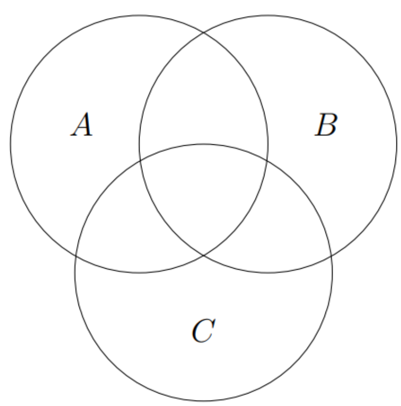
$$|A \cup B| = |A| + |B| - |A \cap B|. \tag{3}$$

Problem 2.17. *Prove (3).*

Problem 2.18. *In a class of 40 students, 25 of them love ice-cream and 22 of them love cheesecake. 16 of the them love both ice-cream and cheesecake. How many students love neither?*

Problem 2.19. *Prove the following statements: (You can use the Venn diagram below as an illustration, then show it algebraically.)*

1. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
2. $(A \cap C) \cup B = (A \cup B) \cap (B \cup C)$



Problem 2.20. *Demonstrate the following inclusion-exclusion principle using a Venn diagram, then prove it algebraically:*

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Problem 2.21. *Among 20 students in a room, 9 study Mathematics, 10 study Science, and 8 take Music classes. 4 students study Mathematics and Science, 4 take Mathematics and Music, and 3 take Science and Music. 2 students take all the three subjects. How many students take at least one of the three courses? How many take none? Solve the problem in two (slightly) different ways, using a Venn diagram and using the inclusion-exclusion principle.*

Problem 2.22. *How many integers between 1 and 100 are not divisible by either* $2, 3,$ *or* 5*?*