Olympiads: Inequalities

ORMC

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Remember to *write down your solutions, as proofs.* You don't have to start by writing out a full proof to every problem you try, but once you've solved a problem or two, take a few minutes to write out a proof as if this was being graded at an Olympiad.

1 Squares!

When in doubt with inequalities, see if you can make something into a sum of squares, which will be nonnegative, and only 0 when all of the squared things are 0.

Problem 1.1 (Putnam and Beyond)**.** For which positive integers n does

$$nx^4 + 4x + 3 = 0$$

have a real solution?

Problem 1.2 (BAMO 2000 Problem 3)**.** Let $x_1, \dots, x_n$ be positive integers, with $n \geq 2$. Prove that

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\dots\left(x_n + \frac{1}{x_n}\right) \geq \left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\dots\left(x_n + \frac{1}{x_1}\right).$$

Problem 1.3 (Putnam and Beyond)**.** Let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n \geq n^2$ and $a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^3 + 1$. Prove that $n - 1 \leq a_k \leq n + 1$ for all k.

Problem 1.4. Prove a simple version of AM-GM: if a, b are nonnegative reals, then $\frac{a+b}{2} \geq \sqrt{ab}$.

Problem 1.5. Prove the *Cauchy-Schwarz* Inequality:

If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are real numbers, then

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Also, under what circumstances are these equal?

2 Cauchy-Schwarz

Problem 2.1 (Problem-Solving Through Problems)**.** Let a, b, c be nonnegative reals with $a + b + c = 1$. Show that

$$ab + bc + ca \leq \frac{1}{3}.$$

Problem 2.2 (USAMO 2009 Problem 4)**.** For $n \geq 2$ let a_1, a_2, ..., a_n be positive real numbers such that $(a_1 + a_2 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \leq \left(n + \frac{1}{2}\right)^2$.

Prove that $\max(a_1, a_2, ..., a_n) \leq 4\min(a_1, a_2, ..., a_n)$.

3 AM-GM

Let $a_1, \ldots, a_n$ be nonnegative real numbers. Then their *arithmetic mean* is at least as big as their *geometric mean*:

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \ldots a_n}.$$

If these are equal, then $a_1 = \cdots = a_n$.

The same is true if we use weighted arithmetic and geometric means.

Problem 3.1. Suppose that a, b, c are positive numbers. Prove that:

- $(a+b)(b+c)(c+a) \geq 8abc$.
- $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$.

Problem 3.2 (USAMO 2011 Problem 1)**.** Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + (a+b+c)^2 \leq 4$. Prove that

$$\frac{ab+1}{(a+b)^2} + \frac{bc+1}{(b+c)^2} + \frac{ca+1}{(c+a)^2} \geq 3.$$

Problem 3.3 (Putnam 2021 Problem B2)**.** Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 \ldots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \ldots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$

4 Hölder's Inequality

For this problem, you may want to use a souped-up version of Cauchy-Schwarz called *Hölder's Inequality*. Let $a_{ij} : 1 \leq i \leq m, 1 \leq j \leq n$ and $p_i : 1 \leq i \leq m$ be nonnegative reals with $\sum_{i=1}^{m} p_i = 1$. Then

$$\prod_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right)^{p_i} \geq \sum_{j=1}^{n} \prod_{i=1}^{m} a_{ij}^{p_i}.$$

Problem 4.1 (USAMO 2004 Problem 5)**.** Let a, b, and c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a+b+c)^3.$$

Hint: First prove that $x^5 + 1 \geq x^3 + x^2$.