Isoperimetric Inequality and Random Chords

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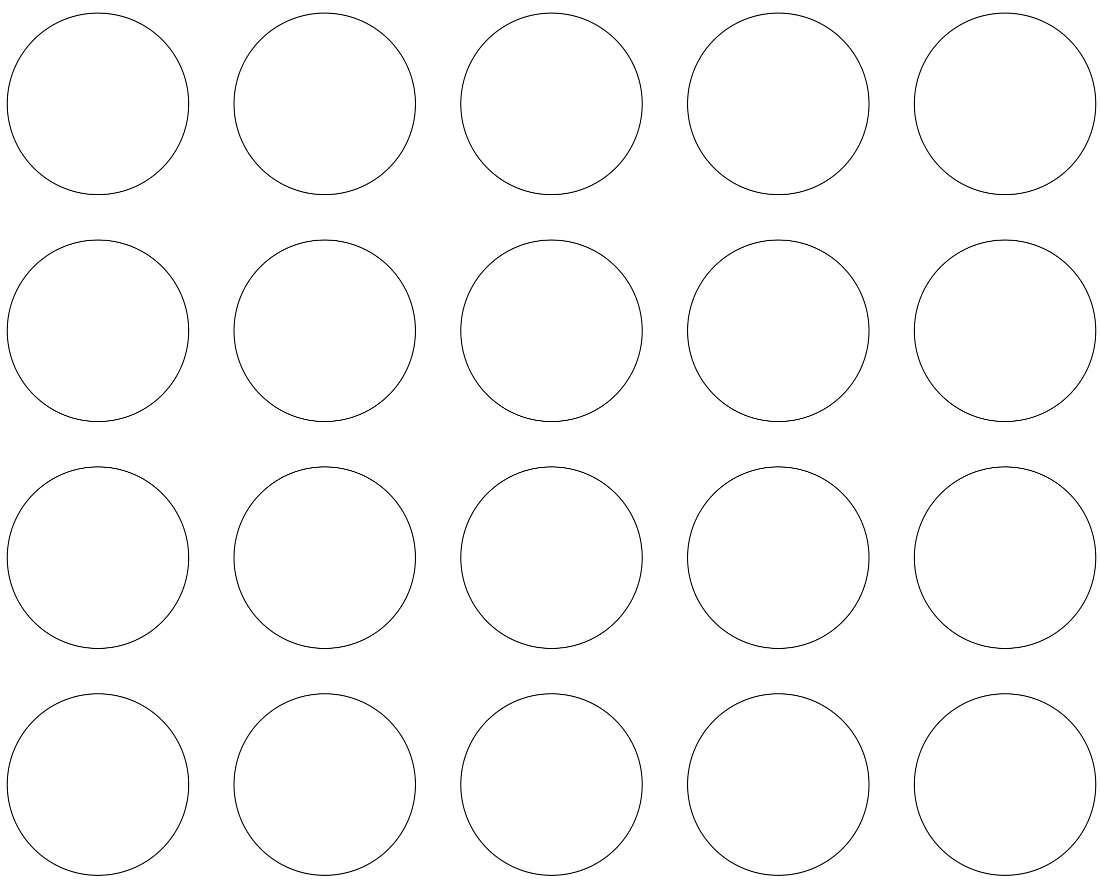
Advanced 1 - Fall 2023

1 Random Chords of a Circle

Last week we proved the *isoperimetric inequality*, which states that the planar region with the most area for its perimeter is a circle. We'll now study a problem that is seemingly unrelated at first, of random chords.

Problem 1 *Given a circle (like the ones drawn below), perform the following experiment:*

1. *Drop a straight object (like a toothpick, pencil, or ruler) onto the circle, which should cut a chord through the circle. (If you missed the circle, redo the drop.)*
2. *Do this again to get another chord, and see if they intersect.*
3. *Repeat Steps 1-2 until you have a good idea of the probability that two random chords of a circle intersect each other. You may check with your classmates if you are unsure.*

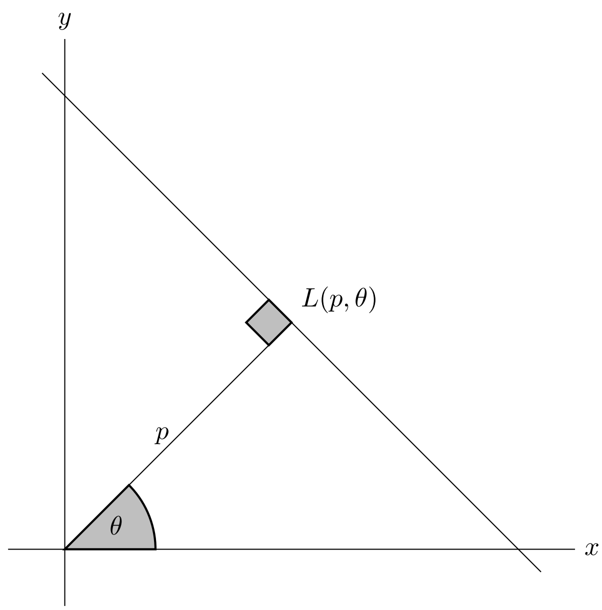


In order to prove our conjecture from Problem 1, we should formalize what it means to take a random chord of a circle. It will be more convenient to take a random line instead, so we'll show that these are in some sense the same.

Problem 2 *Give a one-to-one and onto correspondence between chords of a circle and secant lines (that is, lines that pass through the interior) of that circle.*

Solution: Extend a chord to a secant line, or take the part of the secant line inside the circle for a chord.

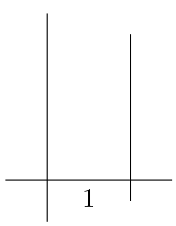
When we did the experiment in Problem 1, we (in theory) dropped a line with a random angle and a random distance from the center of the circle. So to take random chords of a circle, we ought to control these parameters. Formally, the line $L(p, \theta)$ is the line in the xy-plane whose perpendicular from the origin has length p and makes angle θ (in radians) with the x-axis, as shown in the diagram below.



Problem 3 *Draw the following lines in the xy-plane (continued on the following page).*

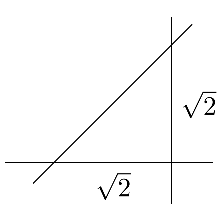
- $L(1, 0)$

Solution:

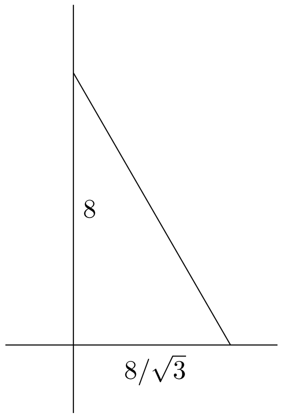


- $L(1, 3\pi/4)$

Solution:



- $L(4, \pi/6)$



Solution:

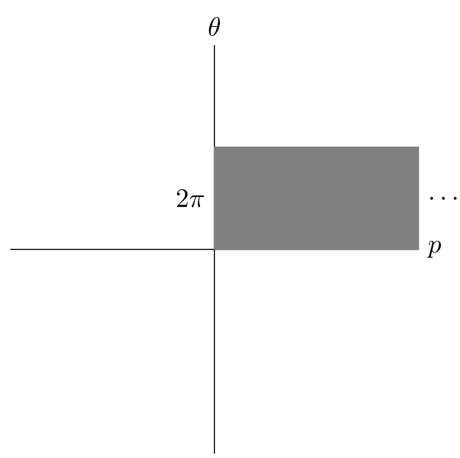
- $L(0, \pi/2)$

Solution: The x-axis.

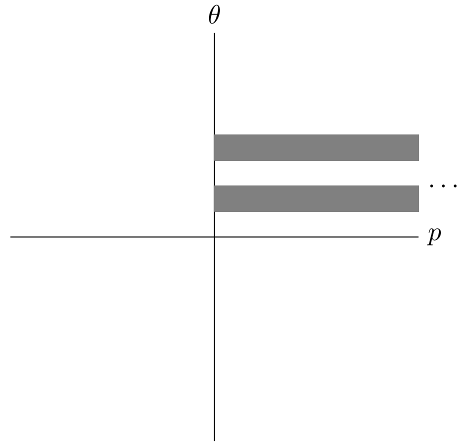
- $L(0, \pi)$

Solution: The y-axis

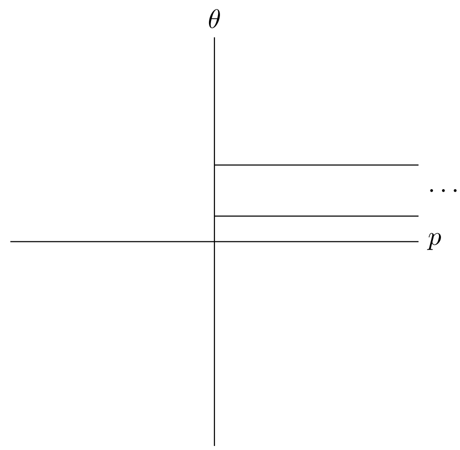
Problem 4 *The following axes give the "(p, θ)-plane". Draw the region that contains all possible values of (p, θ) for a line in the xy-plane.*



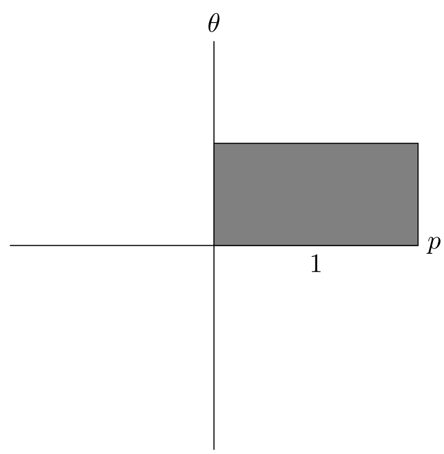
Problem 5 *In the (p, θ)-plane below, draw the region that contains the possible (p, θ) for lines in the xy-plane with positive slope. Also justify the drawing.*



Problem 6 *In the (p, θ)-plane below, draw the region that contains the possible (p, θ) for lines in the xy-plane parallel to the x-axis. Also justify the drawing.*



Problem 7 *In the (p, θ)-plane below, draw the region that contains the possible (p, θ) for a chord of the unit circle (the circle of radius 1 centered at the origin). Also justify the drawing.*



We'll say that the *amount of lines* with a certain property is the area of the region that contains the possible (p, θ) for lines with that property.

Problem 8 *Find the amount of lines in the xy-plane that*

- *have positive slope.*

  Solution: ∞

- *are parallel to the x-axis.*

  Solution: 0

- *give chords of the unit circle.*

  Solution: 2π

- *are tangent to the unit circle.*

  Solution: 0

We are now almost ready to compute probabilities involving lines. Recall that for a finite set, a *sample space* Ω is a set that contains all possible outcomes of an experiment, and an *event* E is any subset of the sample space. The definitions for random lines are similar.

Definition 1
- *A* ***sample space*** Ω *of lines is a subset of the region drawn in Problem 4 that has a well-defined finite area.*
- *Given a sample space* Ω*, an* ***event*** *is a subset* $E \subseteq \Omega$ *with a well-defined area.*
- *Given a sample space* Ω *and an event* E*, we say that the probability of event* E *in* Ω *is*

$$\mathbb{P}(E) := \frac{\textit{area}(E)}{\textit{area}(\Omega)} = \frac{\textit{amount of lines in } E}{\textit{amount of lines in } \Omega}$$

Problem 9 *Find the amount of lines with* $p < 1$ *that have positive slope. Then find the probability that a random line with* $p < 1$ *has positive slope. (Hint: First figure out the sample space.)*

Solution: The amount of lines with $p < 1$ is 2π, and the amount with positive slope is π, so the probability is $1/2$.

Problem 10 *Find the amount of lines with $p < 1$ that are parallel to the x-axis. Then find the probability that a random line with $p < 1$ is parallel to the x-axis.*

Solution: 0 (by Problem 8)

Problem 11 *Find the amount of lines with $p < 1$ that are chords of the unit circle. Then find the probability that a random line with $p < 1$ is a chord of the unit circle.*

Solution: 1, since every line with $p < 1$ is a chord of the unit circle. Alternatively, the amount of both kinds of lines is 2π.

We'll now check our answer from Problem 1. Since all circles are similar, assume for the sake of simplicity that all circles in the next problem are the unit circle (centered at the origin).

Problem 12

- *Consider the x-axis, which intersects the unit circle in a diameter (and in particular, a chord). Find the region in (p, θ) that contains all the chords of the unit circle that intersect the x-axis inside the unit circle.*

  Solution: The x-intercept of $L(p, \theta)$ is exactly $p \cos\theta$ - in particular, this region is symmetric about $\theta = \pi/2$.

- *Find the amount of chords of the unit circle that intersect the x-axis inside the unit circle. Then show that a random chord of the unit circle intersects the x-axis inside the unit circle with probability $1/2$.*

  Solution: By symmetry, it's π. Since there are 2π many chords of the unit circle, the probability that a random chord intersects the diamter is $1/2$.

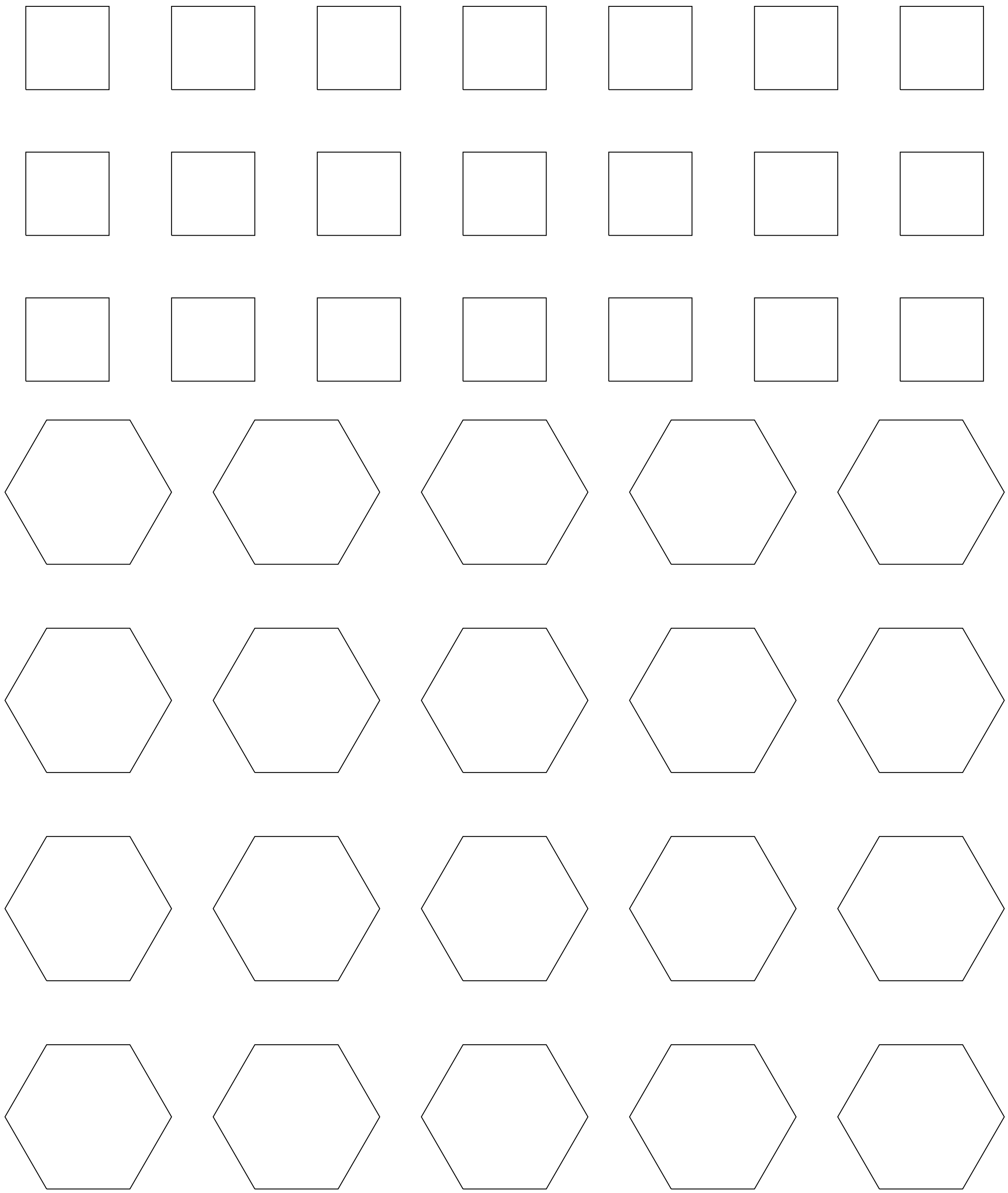
- *Fix any other chord of the unit circle, and show that a random chord of the unit circle intersects it inside the unit circle with probability $1/2$. (Hint: By rotating, we can assume the fixed chord is parallel to the x-axis.) Conclude that the probability that any two chords of a circle intersect (inside the circle) is $1/2$.*

  Solution: Similar to the first two parts.

2 Intersections and Crofton's Formula

Last week, we showed that given shapes of the same perimeter, a circle has the most area, followed by a regular hexagon, followed by a square. We would now like to study random chords in each of these shapes – in general, a *chord* of a convex region is a line segment that begins and ends on the edge of the region.

Problem 13 *Repeat the experiment from Problem* 1 *on the following squares and hexagons. Which do you think has a higher probability, and what about compared to the circle?*



While we could do a similar analysis as in Problem 12, the lack of rotational symmetry will make things much more complicated. Instead, to prove the pattern shown in Problem 13, we use the following formula.

Theorem 1 *(Crofton's Formula) Suppose a curve C in the xy-plane has length l. Then*

$$\begin{aligned} 2l = 1 \times \ & \textit{amount of lines that intersect } C \textit{ once} \\ + 2 \times \ & \textit{amount of lines that intersect } C \textit{ twice} \\ + 3 \times \ & \textit{amount of lines that intersect } C \textit{ thrice} \\ + \dots & \end{aligned}$$

Problem 14 *Verify that Crofton's Formula works in the case that C is a circle.*

Solution: By Problem 8, there is a zero amount of lines that are tangent to C. Every other line either intersects C 0 or 2 times, and the ones that intersect it twice are exactly the chords. Similarly to Problem 7, there are $2\pi r$ chords of the circle C with radius r, so we do indeed have

$$2(2\pi r) = 1 \times 0 + 2 \times (2\pi r) + 3 \times 0 + 4 \times 0 + ...$$

Problem 15 *(Bonus) Prove Crofton's Formula. (Hint: The expression on the right-hand side of the formula looks a lot like the expression for expected value. Think about how we solved Buffon's needle problem!)*

Solution: If C can be broken into two pieces C_1 and C_2, then by a similar argument as linearity of expectation, Crofton's formula holds for C if and only if it holds for C_1 and C_2. Now break a line segment into many pieces and rearrange them to approximate a circle. By Problem 14, we know Crofton's formula holds for the circle, so it must hold for the line segment in the first place. Now given any curve, we can approximate it with some line segments, which proves it for the curve by the same logic.

As a reminder, the Isoperimetric Inequality can be written in equation form as follows:

Theorem 2 *(Isoperimetric Inequality) Let R be a region in the plane with area A and perimeter P. Then*

$$4\pi A \leq P^2$$

with equality if and only if R is the inside of a circle.

Problem 16 *Find the "amount" of pairs of chords of a convex region that intersect, in terms of its area. (Hint: "How many" chords intersect at any given point? The answer won't be an integer.)*

Solution: Given an ordered pair of chords C_1, C_2, there are 2π possible angles between C_1 and C_2, so there are 2π chords that intersect at any given point, and thus $2\pi A$ chords that intersect total.

Problem 17 *Last week, we showed it suffices to assume the region is convex. How many times can a line intersect the perimeter of a convex region?*

Solution: Twice, by the same logic as last week. (Tangent with probability 0)

Problem 18 *Recall the* ***isoperimetric quotient*** *is the quantity $Q = 4\pi A/P^2$. Using Crofton's Formula, prove that*

$$\mathbb{P}(\textit{two chords of } R \textit{ intersect}) = \frac{1}{2}Q$$

Solution: By Crofton's Formula and since all chords intersect the perimeter twice,

$$2P = 2 \times \text{amount of chords of } R$$

so that P is the amount of chords of R, and hence P^2 is the amount of ordered pairs of chords of R. But then

$$\mathbb{P}(\text{two chords of } R \text{ intersect}) = \frac{\text{amount of intersecting chords}}{\text{amount of pairs of chords}} = \frac{2\pi A}{P^2} = \frac{1}{2}Q$$

Problem 19 *Use the previous problem to show the following probabilistic form of the Isoperimetric Inequality:*

Theorem 3 *Given a convex region R in the plane,*

$$\mathbb{P}(\textit{two chords of } R \textit{ intersect}) \leq \frac{1}{2}$$

with equality if and only if R is the inside of a circle.

Solution: This is immediate from the previous problem.

We can now also experimentally calculate the area of a figure, by repeating the process of Problems 1 and 13.

Problem 20 *(Bonus) Design an experiment to calculate the area of each figure below using only a ruler and (possibly) some string. If you have time, try doing some of the experiments.*