OLGA RADKO MATH CIRCLE: ADVANCED 3

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Worksheet 6: Rings II

Let $(R, +, \times)$ be a commutative ring. An *ideal* of R is a subset I of R, that satisfies the following properties:

(1) 0 is in I.
(2) If a is an element in I, then $-a$ is also in I.
(3) If a and b are elements in I, then $a + b$ is also in I.
(4) For any a in I and r in R, the element $r \times a$ is in I

In this worksheet all the rings will be assumed to be commutative.

Problem 6.0: Determine which of the following subsets are ideals of the corresponding ring:

- The even integers in $\mathbb{Z}$
- The "even integers" $\{0, 2, 4\}$ in $\mathbb{Z}/6\mathbb{Z}$
- The "even integers" $\{0, 2, 4\}$ in $\mathbb{Z}/5\mathbb{Z}$
- The positive numbers in $\mathbb{Q}$

Solution 6.0:

Problem 6.1: Show that for any ring R the sets $\{0\}$ and R are ideals. Can you find any ring where these are the only possible ideals?
Solution 6.1:

Let's see a couple of examples of rings that lie in between other rings that we have worked with:

Example 1: An example of a subring of $\mathbb{C}$ is the set of elements of the form $a + bi$, where a and b are integers. This ring is called the *Gaussian integers*, and is denoted as $\mathbb{Z}[i]$.

Example 2: We define $\mathbb{Z}[\sqrt{2}]$ the subring of $\mathbb{R}$ to be formed by the elements of the form $a + b\sqrt{2}$, where a and b are integers.

Problem 6.2: Notice that both $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[i]$ have $\mathbb{Z}$ as a subring. Is this an ideal of any of those rings?

Can you explain some differences between subrings and ideals?

Solution 6.2:

Let's remember examples of rings from last week.

Example 1: If $(R, +_R, \times_R)$ and $(S, +_S, \times_S)$ are two rings, then we define the operations in the ring $R \times S$ as follows:

- Its elements are of the form (a, b) for any a in R and b in S.
- Addition is performed term by term, i.e. $(a, b) + (c, d) = (a +_R c, b +_S d)$.
- Multiplication is performed term by term i.e. $(a, b) \times (c, d) = (a \times_R c, b \times_S d)$.

Example 2: If R is a ring we can define the ring of polynomials $R[x]$. Whose elements are polynomials in one variable with coefficients in R. Addition is performed by adding up the coefficients of monomials of the same degree, while multiplication is performed by multiplying the coefficients in the field and adding up the degrees of the monomials x^i, respecting the distributive property.

Problem 6.3: Find 4 different ideals in $\mathbb{Z} \times \mathbb{Z}$.

Are all of the ideals of $\mathbb{Z} \times \mathbb{Z}$ of the form $I \times J$, where I and J are ideals of $\mathbb{Z}$.

Solution 6.3:

Problem 6.4: Show that in $\mathbb{Z}/6\mathbb{Z}$ you have 4 possible ideals, and that they "correspond" to the divisors of 6. Can you see how the ideals of $\mathbb{Z}/m\mathbb{Z}$ look in general?

Solution 6.4:

Problem 6.5: What are the ideals of $\mathbb{Z}$?
Solution 6.5:

Problem 6.6: Show that any field F has only two ideals $\{0\}$ and F.
Solution 6.6:

In the ring $\mathbb{Z}/2\mathbb{Z}$, the ideal $\{0\}$ contains almost all the elements. It is only missing a single element of the ring.

Problem 6.7: Are there any other rings R that have an ideal I that contains all of the elements of R except for a single element?

How many examples of rings R with an ideal I that misses exactly two elements of R can you find?

Solution 6.7:

Remember that an element a is a zero-divisor if there exists some non-zero element b such that $a \times b = 0$.

Problem 6.8: Give examples of rings with non-zero zero divisors, where the set of all zero divisors (and 0) forms an ideal.

Give examples of rings, where the set of all zero divisors (and 0) does not form an ideal.

Solution 6.8:

An element a in a ring R is called nilpotent if some power of it is 0, i.e. for some integer n, we have $a \times \ldots \times a = a^n = 0$.

For example, 0 is a nilpotent element in any ring and 2 is a nilpotent element in $\mathbb{Z}/4\mathbb{Z}$.

Problem 6.9: Show that the set of nilpotent elements forms an ideal

Solution 6.9:

Problem 6.10:

(1) Show that if I is an ideal in R, and J is an ideal in S. Then $I \times J$ is an ideal in $R \times S$.
(2) Let R and S be rings. Prove that any ideal in $R \times S$ is of the form $I \times J$, where I is an ideal in R and J is an ideal in S.

Solution 6.10:

Problem 6.11: Can you find a ring R with a subring containing all but a single element of R? What about a subring missing exactly 2 elements?

How many such rings and subrings can you find? **Solution 6.11:**

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