Sequences and Series

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October 29, 2023

1 Warm Up

The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 33 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

(A) $\frac{1}{21}$ (B) $\frac{1}{4}$ (C) $\frac{5}{63}$ (D) $\frac{2}{21}$ (E) $\frac{1}{7}$

How many ways can a student schedule 3 mathematics courses – algebra, geometry, and number theory – in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

(A) 3 (B) 6 (C) 12 (D) 18 (E) 24

For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

(A) 14 (B) 16 (C) 18 (D) 19 (E) 21

A child builds towers using identically shaped cubes of different colors. How many different towers with a height 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

(A) 24 (B) 288 (C) 312 (D) 1260 (E)4032

2 Arithmetic Sequences and Series

2.1 Arithmetic Sequences

What is an arithmetic sequence? An arithmetic sequence is a sequence where the differences between the terms are all equal.

For example, the sequence 2, 5, 8, 11... is an arithmetic sequence because all the terms have a difference of 3!

Now, there are two ways to represent this sequence, we can say the first number of the sequence has an index of 1 or 0.

If we are interested in the n-th term of a progression, where n is a natural number, we can call it a_n and write it as $a_n = d * (n-1) + a_1$, where d is the difference between consecutive terms in our progression

2.1.1 Example

15 children were gathering mushrooms. Together, the children gathered 100 mushrooms. Show that at least two children picked the same number of mushrooms.

2.2 Arithmetic Series

Series: A sum of a certain number of terms in a sequence

Let's take a look at the summation notation for an arithmetic series

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + ... + f(b-1) + f(b)$$

You are essentially adding all the terms between index a and b.

To find the sum of an arithmetic sequence, use this formula:

$$\frac{(f(a)+f(b))(a-b+1)}{2}$$

$(a-b+1)$ is the number of terms between a and b.

2.2.1 Examples

For the sequences listed below find the:

1. Equation using zero indexing
2. Series
3. The next term if the sequence continues

(1) 6, 12, 18, 24, 30, 36, 42, 48

(2) 4, 7, 10, 13, 16, 19, 22, 25

2.3 Additional Properties of Arithmetic Sequences

There are some helpful properties of arithmetic series that you should remember. These tricks will help you break apart larger series.

1. $x * \sum_{i=a}^{b} f(a) = \sum_{i=a}^{b} x * f(a)$
2. $\sum_{i=a}^{b} f(i) + \sum_{i=a}^{b} g(i) = \sum_{i=a}^{b} \big(f(i) + g(i)\big)$

3 Geometric Sequences and Series

3.1 Geometric Sequences

Geometric Sequences are a bit different from arithmetic sequences because the terms do not share a common difference. Instead, each term has a common quotient. We can represent these sequences like so:

$$a_n = q \cdot r^n$$

where q and r are constants.

For example, $3, 6, 12, 24, 48$ is a geometric sequence, where q is 3 and r is 2. (Remember we are using zero indexing!) Each term has a common quotient of 2.

3.1.1 Example

Write out the first 5 terms of a_n, where $a_n = 5(3)^n$

3.2 Geometric Series

Like an arithmetic series, a geometric series is the sum of a certain number of terms in a geometric sequence. You can also add infinite terms of geometric series in certain situations. However, right now we will work on deriving the formula for the sum of a finite geometric series.

A finite geometric series is one where the number of values in the sequence is known to us, and thus we can add them all up to get our sum. Using this information, start with the below to try and derive the formula for the sum of a finite geometric sequence.

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}$$

Next, we can look at the sum of an infinite geometric sequence. Say we have an infinite geometric series whose first term is a and common ratio is r. If r is between -1 and 1 (i.e. $|r| < 1$), then the series converges into the following finite value:

$$\sum_{i=0}^{\infty} a \cdot r^i = \frac{a}{(1-r)}$$

Reminder that this only works when $|r| < 1$! This is because when you start finding enough terms, the difference between them is so minimal that adding them also has minimal effect, and so the sum will continue towards one number infinitely.

One of the most famous geometric sequences is $(\frac{1}{2})^n$. Find the sum of this infinite geometric sequence.

3.2.1 Examples

For the sequences listed below find the:

1. Equation using zero indexing
2. Series
3. The next term if the sequence would continue

(1) 7, 35, 175, 875, 4375

(2) 9, 36, 144, 576, 2304

4 Practice

For each positive integer k, let $S(k)$ denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is k. For example, $S3$ is the sequence $1, 4, 7, 10, ...$ For how many values of k does $S(k)$ contain the term 2005?

The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the original sequence's first, last, and middle terms.

The first three terms of a geometric progression are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$. What is the fourth term?

Let $a < b < c$ be three integers such that a, b, c is an arithmetic progression and a, c, b is a geometric progression. What is the smallest possible value of c?

Each row and each column in this 5x5 array is an arithmetic sequence with five terms. What is the value of X?

1				25
		X		
17				81

(A) 21 (B) 31 (C) 36 (D) 40 (E) 42

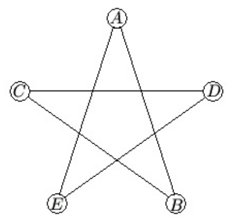
How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?

(A) 0 **(B)** 1 **(C)** 59 **(D)** 89 **(E)** 178

A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains 100 cans, how many rows does it contain?

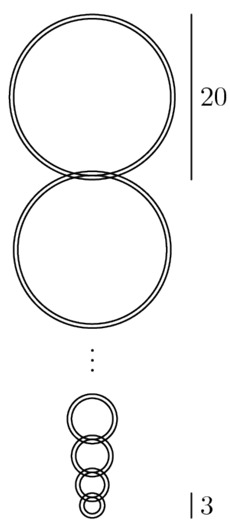
(A) 5 (B) 8 (C) 9 (D) 10 (E) 11

In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers $3, 5, 6, 7$, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments AB, BC, CD, DE, and EA form an arithmetic sequence, although not necessarily in that order. What is the middle term of the arithmetic sequence?



(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?



(A) 171 **(B)** 173 **(C)** 182 **(D)** 188 **(E)** 210