Continued Fractions

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Based on a handout by Matthew Gherman and Adam Lott

Part 1: The Euclidean Algorithm

Definition 1:
The *greatest common divisor* of a and b is the greatest integer that divides both a and b.
We denote this number with $\gcd(a, b)$. For example, $\gcd(45, 60) = 15$.

Problem 2:
Find $\gcd(20, 14)$ by hand.

Theorem 3: The Division Algorithm
Given two integers a, b, we can find two integers q, r, where $0 \leq r < b$ and $a = qb + r$.
In other words, we can divide a by b to get q remainder r.

For example, take $14 \div 3$. We can re-write this as $3 \times 4 + 2$.
Here, a and b are 14 and 3, $q = 4$ and $r = 2$.

Theorem 4:
For any integers a, b, c,
$\gcd(ac + b, a) = \gcd(a, b)$

Problem 5:
Compute $\gcd(668, 6)$ *Hint:* $668 = 111 \times 6 + 2$
Then, compute $\gcd(3 \times 668 + 6, 668)$.

Problem 6: The Euclidean Algorithm
Using the two theorems above, detail an algorithm for finding $\gcd(a, b)$.
Then, compute $\gcd(1610, 207)$ by hand.

Part 2:

Definition 7:
A *finite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + ... + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}}}$$

where $a_0, a_1, ..., a_k$ are all in $\mathbb{Z}^+$. We'll denote this as $[a_0, a_1, ..., a_k]$.

Problem 8:
Write each of the following as a continued fraction.
Hint: Solve for one a_n at a time.

- 5/12
- 5/3
- 33/23
- 37/31

Problem 9:
Write each of the following continued fractions as a regular fraction in lowest terms:

- $[2, 3, 2]$
- $[1, 4, 6, 4]$
- $[2, 3, 2, 3]$
- $[9, 12, 21, 2]$

Problem 10:
Let $\frac{p}{q}$ be a positive rational number in lowest terms. Perform the Euclidean algorithm to obtain the following sequence:

$$\begin{aligned} p &= q_0 q + r_1 \\ q &= q_1 r_1 + r_2 \\ r_1 &= q_2 r_2 + r_3 \\ &\vdots \\ r_{k-1} &= q_k r_k + 1 \\ r_k &= q_{k+1} \end{aligned}$$

We know that we will eventually get 1 as the remainder because p and q are relatively prime.
Show that $p/q = [q_0, q_1, ..., q_{k+1}]$.

Problem 11:
Repeat Problem 8 using the method outlined in Problem 10.

Definition 12:
An *infinite continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + ...}}}}$$

where $a_0, a_1, a_2, ...$ are in $\mathbb{Z}^+$. To prove that this expression actually makes sense and equals a finite number is beyond the scope of this worksheet, so we assume it for now. This is denoted $[a_0, a_1, a_2, ...]$.

Problem 13:
Using a calculator, compute the first five terms of the continued fraction expansion of the following numbers. Do you see any patterns?

- $\sqrt{2}$
- $\pi \approx 3.14159...$
- $\sqrt{5}$
- $e \approx 2.71828...$

Problem 14:
Show that an $\alpha \in \mathbb{R}^+$ can be written as a finite continued fraction if and only if α is rational.
Hint: For one of the directions, use Problem 10

Definition 15:
The continued fraction $[a_0, a_1, a_2, ...]$ is *periodic* if it ends in a repeating sequence of digits. A few examples are below. We denote the repeating sequence with a line.

- $[1, 2, 2, 2, ...] = [1, \overline{2}]$ is periodic.
- $[1, 2, 3, 4, 5, ...]$ is not periodic.
- $[1, 3, 7, 6, 4, 3, 4, 3, 4, 3, ...] = [1, 3, 7, 6, \overline{4, 3}]$ is periodic.
- $[1, 2, 4, 8, 16, ...]$ is not periodic.

Problem 16:

- Show that $\sqrt{2} = [1, \overline{2}]$.
- Show that $\sqrt{5} = [1, \overline{4}]$.

Hint: use the same strategy as Problem 13 but without a calculator.

Problem 17: Challenge I
Express the following continued fractions in the form $\frac{a+\sqrt{b}}{c}$ where a, b, and c are integers:

- $[\ \overline{1}\ ]$
- $[\ \overline{2, 5}\ ]$
- $[\ 1, 3, \overline{2, 3}\ ]$

Problem 18: Challenge II
Let $\alpha = [\ a_0,\ ...,\ a_r,\ \overline{a_{r+1},\ ...,\ a_{r+p}}\ ]$ be any periodic continued fraction.
Prove that α is of the form $\frac{a+\sqrt{b}}{c}$ for some integers a, b, c where b is not a perfect square.

Problem 19: Challenge III
Prove that any number of the form $\frac{a+\sqrt{b}}{c}$ where a, b, c are integers and b is not a perfect square can be written as a periodic continued fraction.

Part 3: Convergents

Definition 20:
Let $\alpha = [a_0, a_1, a_2, ...]$ be an infinite continued fraction (aka an irrational number).
The *nth convergent to* α is the rational number $[a_0, a_1, ..., a_n]$ and is denoted $C_n(\alpha)$.

Problem 21:
Calculate the following convergents and write them in lowest terms:

- $C_3([\ 1, 2, 3, 4, ...\ ])$
- $C_4([\ 0, \overline{2, 3}\ ])$
- $C_5([\ \overline{1, 5}\ ])$

Problem 22:
Recall from last week that $\sqrt{5} = [2, \overline{4}]$. Calculate the first five convergents to $\sqrt{5}$ and write them in lowest terms. Do you notice any patterns?
Hint: Look at the numbers $\sqrt{5} - C_j(\sqrt{5})$ for $1 \leq j \leq 5$

Properties of Convergents
In this section, we want to show that the nth convergent to a real number α is the best approximation of α with the given denominator. Let $\alpha = [a_0, a_1, ...]$ be fixed, and we will write C_n instead of $C_n(\alpha)$ for short. Let p_n/q_n be the expression of C_n as a rational number in lowest terms. We will eventually prove that $|\alpha - C_n| < \frac{1}{q_n^2}$, and there is no better rational estimate of α with denominator less than or equal to q_n.
First we want the recursive formulas $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ given $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, and $q_0 = 1$.

Problem 23:
Verify the recursive formula for $1 \leq j \leq 3$ for the convergents C_j of:

- $[\ 1, 2, 3, 4, ...\ ]$
- $[\ 0, \overline{2, 3}\ ]$
- $[\ \overline{1, 5}\ ]$

Problem 24: Challenge IV
Prove that $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ by induction.

- As the base case, verify the recursive formulas for $n = 1$ and $n = 2$.
- Assume the recursive formulas hold for $n \leq m$ and show the formulas hold for $m + 1$.

Problem 25:
Using the recursive formula from Problem 24, we will show that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

- What is $p_1 q_0 - p_0 q_1$?
- Substitute $a_n p_{n-1} + p_{n-2}$ for p_n and $a_n q_{n-1} + q_{n-2}$ for q_n in $p_n q_{n-1} - p_{n-1} q_n$. Simplify the expression.
- What happens when $n = 2$? Explain why $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

Problem 26: Challenge VI
Similarly derive the formula $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-2} a_n$.

Problem 27:
Recall $C_n = p_n/q_n$. Show that $C_n - C_{n-1} = \frac{(-1)^{n-1}}{q_{n-1}q_n}$ and $C_n - C_{n-2} = \frac{(-1)^{n-2}a_n}{q_{n-2}q_n}$. *Hint:* Use Problem 25 and $p_nq_{n-2} - p_{n-2}q_n = (-1)^{n-2}a_n$ respectively
In Problem 22, the value $\alpha - C_n$ alternated between negative and positive and $|\alpha - C_n|$ got smaller each step. Using the relations in Problem 27, we can prove that this is always the case. Specifically, α is always between C_n and C_{n+1}.

Problem 28:
Let's figure out how well the nth convergents estimate α. We will show that $|\alpha - C_n| < \frac{1}{q_n^2}$.

- Note that $|C_{n+1} - C_n| = \frac{1}{q_nq_{n+1}}$.
- Why is $|\alpha - C_n| \leq |C_{n+1} - C_n|$?
- Conclude that $|\alpha - C_n| < \frac{1}{q_n^2}$.

We are now ready to prove a fundamental result in the theory of rational approximation.

Problem 29: Dirichlet's approximation theorem
Let α be any irrational number. Prove that there are infinitely many rational numbers $\frac{p}{q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.

Problem 30: Challenge VII
Prove that if α is *rational*, then there are only *finitely* many rational numbers $\frac{p}{q}$ satisfying $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.
The above result shows that the nth convergents estimate α extremely well. Are there better estimates for α if we want small denominators? In order to answer this question, we introduce the Farey sequence.

Definition 31:
The *Farey sequence* of order n is the set of rational numbers between 0 and 1 whose denominators (in lowest terms) are $\leq n$, arranged in increasing order.

Problem 32:
List the Farey sequence of order 4. Now figure out the Farey sequence of order 5 by including the relevant rational numbers in the Farey sequence of order 4.

Problem 33:
Let $\frac{a}{b}$ and $\frac{c}{d}$ be consecutive elements of the Farey sequence of order 5. What does $bc - ad$ equal?

Problem 34: Challenge VIII
Prove that $bc - ad = 1$ for $\frac{a}{b}$ and $\frac{c}{d}$ consecutive rational numbers in Farey sequence of order n.

- In the plane, draw the triangle with vertices (0,0), (b, a), (d, c). Show that the area A of this triangle is $\frac{1}{2}$ using Pick's Theorem. Recall that Pick's Theorem states $A = \frac{B}{2} + I - 1$ where B is the number of lattice points on the boundary and I is the number of points in the interior.
  Hint: B=3 and I=0
- Show that the area of the triangle is also given by $\frac{1}{2}|ad - bc|$.
- Why is $bc > ad$?
- Conclude that $bc - ad = 1$.

Problem 35:

Use the result of Problem 34 to show that there is no rational number between C_{n-1} and C_n with denominator less than or equal to q_n. Conclude that if a/b is any rational number with $b \leq q_n$, then $|\alpha - \frac{a}{b}| \geq |\alpha - \frac{p_n}{q_n}|$

Problem 36: Challenge V

Prove the following strengthening of Dirichlet's approximation theorem. If α is irrational, then there are infinitely many rational numbers $\frac{p}{q}$ satisfying $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$.

- Prove that $(x+y)^2 \geq 4xy$ for any real x, y.
- Let p_n/q_n be the nth convergent to α. Prove that

$$|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}|^2 \;\geq\; 4|\frac{p_n}{q_n} - \alpha||\frac{p_{n+1}}{q_{n+1}} - \alpha|$$

  Hint: α lies in between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$

- Prove that either $\frac{p_n}{q_n}$ or $\frac{p_{n+1}}{q_{n+1}}$ satisfies the desired inequality (Hint: proof by contradiction).
- Conclude that there are infinitely many rational numbers $\frac{p}{q}$ satisfying $|\alpha - \frac{p}{q}| < \frac{1}{2q^2}$.