ORMC AMC 10/12 Group
Week 5: Triangles

October 29, 2023

1 Warm-up Exercises

1. **(2013 AMC 12A # 1)** Square $ABCD$ has side length 10. Point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40. What is BE?

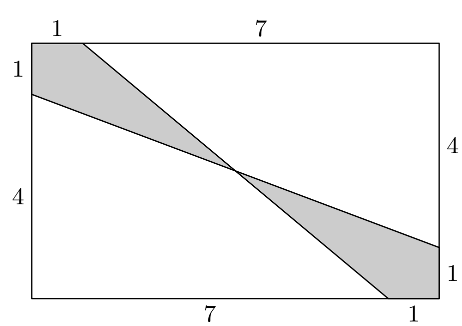
2. **(2015 AMC 12A # 2)** Two of the three sides of a triangle are 20 and 15. What is the range of possible values for the third side?

3. **(2020 AMC 10B # 4)** The acute angles of a right triangle are a° and b°, where $a > b$ and both a and b are prime numbers. What is the least possible value of b?

4. **(2020 AMC 10B # 8)** Points P and Q lie in a plane with $PQ = 8$. How many locations for point R in this plane are there such that the triangle with vertices P, Q, and R is a right triangle with area 12 square units?

5. **(2016 AMC 10B # 9)** All three vertices of $\triangle ABC$ lie on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC?

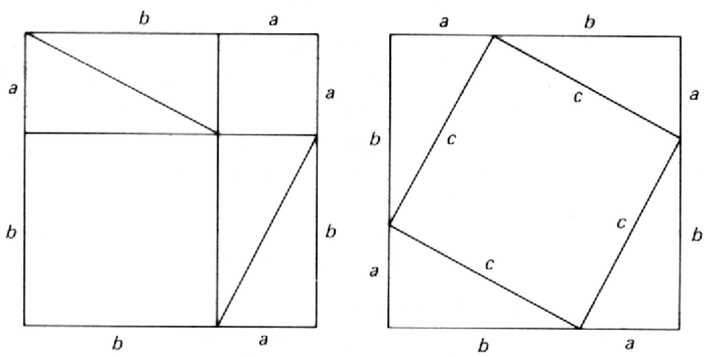
6. **(2016 AMC 10A # 11)** Find the area of the shaded region.



2 Important Facts

A triangle is a right triangle when one of its internal angles has measure 90°.

The ***Pythagorean Theorem*** states that a triangle is a right triangle if and only if its sides a, b, c, WLOG $a \leq b \leq c$, satisfy $a^2 + b^2 = c^2$. This is usually proved using the following type of diagram:



Integer triples a, b, c satisfying $a^2 + b^2 = c^2$ are called **pythagorean triples**; here are a few to remember:

$$(3, 4, 5) \quad (5, 12, 13) \quad (7, 24, 25) \quad (8, 15, 17) \quad (9, 40, 41) \quad (11, 60, 61) \quad (12, 35, 37)$$

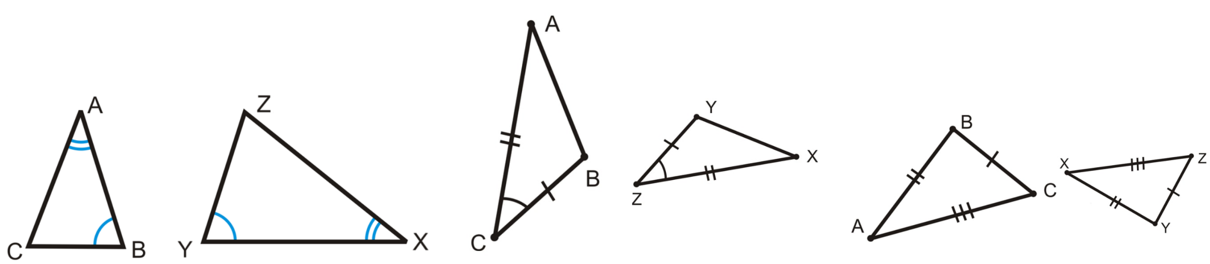
The Pythagorean theorem also gives us some special right triangles that are important to know:

- Isosceles Right Triangle (45-45-90): the sides are in the ratio $1 : 1 : \sqrt{2}$, since we have $a^2 + a^2 = c^2$
- 30-60-90 Right Triangle : the sides are in the ratio $1 : \sqrt{3} : 2$. (note that $1 < \sqrt{3} < 2$.)

Two triangles are considered ***similar*** when one can be obtained from the other by (uniform) scaling, translation, and rotation. We can think of similar triangles as being the same shape, but not the same size. If triangles ABC and DEF are similar triangles, we write $ABC \sim DEF$.

Two triangles must have the same angles in order to be similar. Since the angles always sum to 180°, it is sufficient for triangles to share *two* angles, since the third will be 180 minus the first two. This is called AA or AAA similarity.

Two other ways to show triangles are similar are SSS and SAS. An "A" means "angles are the same", and an "S" means "sides are in the same proportion." Examples of each similarity theorem are below:

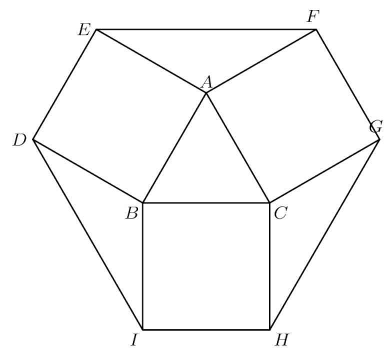


Once you have shown that two triangles are similar using one of the methods written above, the main use of similarity is that *the lengths of corresponding sides are in the same proportion.* For example, in the diagrams above,

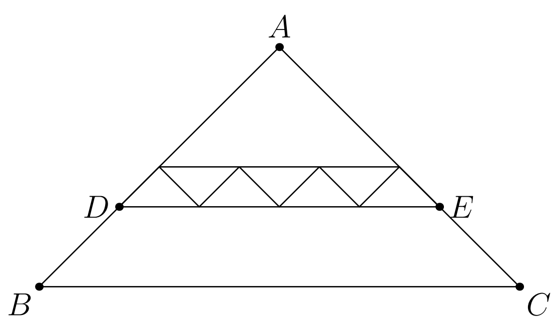
$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}.$$

3 Examples

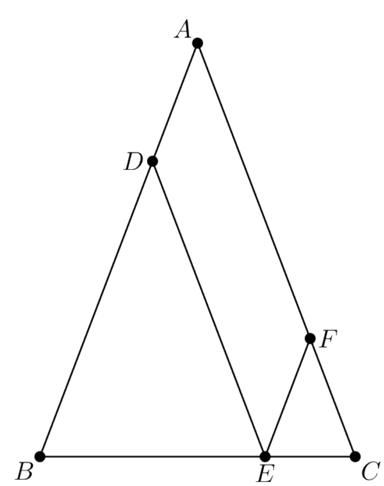
1. Equilateral $\triangle ABC$ has side length 1, and squares $ABDE$, $BCHI$, $CAFG$ lie outside the triangle. What is the area of hexagon $DEFGHI$?



2. All of the triangles in the diagram below are similar to isosceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?

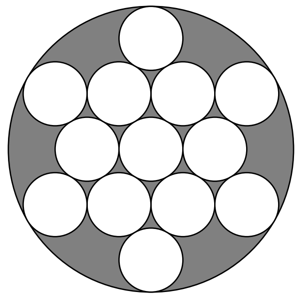


3. **(2013 AMC 12A #9)** In $\triangle ABC$, $AB = AC = 28$ and $BC = 20$. Points D, E, and F are on sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of parallelogram $ADEF$?

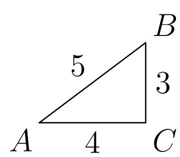


4 Exercises

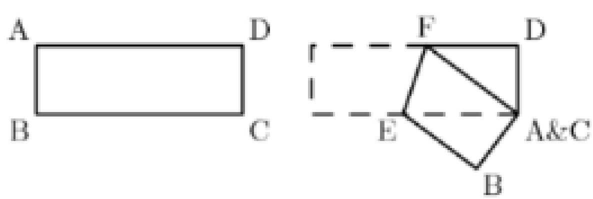
1. **(2019 AMC 10A #16)** The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all the circles of radius 1?



2. **(2018 AMC 10A #13)** A paper triangle with sides of lengths $3, 4,$ and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



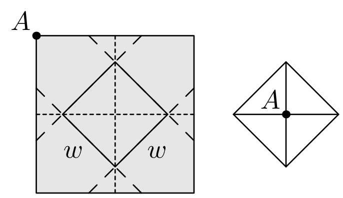
3. In rectangle ABCD, AB = 3 and BC = 9. The rectangle is folded so that points A and C coincide, forming the pentagon ABEFD. What is the length of segment EF? Express your answer in simplest radical form.



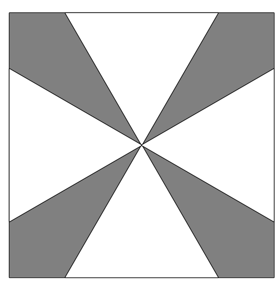
4. **(2019 AMC 10B #15)** Right triangles T_1 and T_2, have areas of 1 and 2, respectively. A side of T_1 is congruent to a side of T_2, and a different side of T_1 is congruent to a different side of T_2. What is the square of the product of the lengths of the other (third) sides of T_1 and T_2?

5. **(2014 AMC 12A #10)** Three congruent isosceles triangles are constructed with their bases on the sides of an equilateral triangle of side length 1. The sum of the areas of the three isosceles triangles is the same as the area of the equilateral triangle. What is the length of one of the two congruent sides of one of the isosceles triangles?

6. **(2018 AMC 10B #15)** A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h. What is the area of the sheet of wrapping paper?



7. **(2019 AMC 10B #8)** The figure below shows a square and four equilateral triangles, with each triangle having a side lying on a side of the square, such that each triangle has side length 2 and the third vertices of the triangles meet at the center of the square. The region inside the square but outside the triangles is shaded. What is the area of the shaded region?

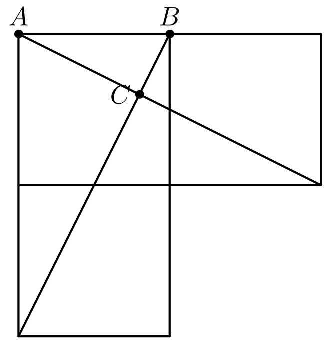


8. **(2015 AMC 10A #19)** The isosceles right triangle ABC has right angle at C and area 12.5. The rays trisecting $\angle ACB$ intersect AB at D and E. What is the area of $\triangle CDE$?

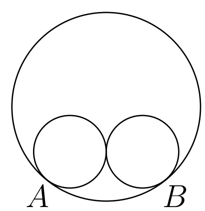
9. **(2021 AMC 10A #13)** What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$?

(Hint: The volume of a pyramid with base area B and height h is $\frac{1}{3}Bh$)

10. **(2012 AMC 10A #15)** Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of $\triangle ABC$?

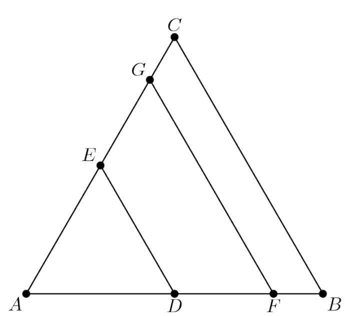


11. **(2018 AMC 10A #15)** Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?



12. **(2018 AMC 10A #16)** Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

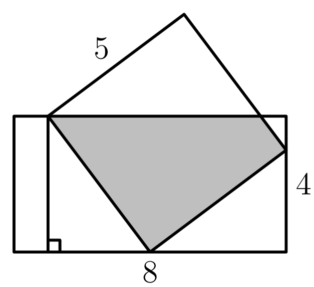
13. **(2013 AMC 12A #11)** Triangle ABC is equilateral with $AB = 1$. Points E and G are on $\overline{AC}$ and points D and F are on $\overline{AB}$ such that both $\overline{DE}$ and $\overline{FG}$ are parallel to $\overline{BC}$. Furthermore, triangle ADE and trapezoids $DFGE$ and $FBCG$ all have the same perimeter. What is $DE + FG$?



14. **(2017 AMC 10B #15)** Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the foot of the perpendicular from B to diagonal $\overline{AC}$. What is the area of $\triangle AED$?

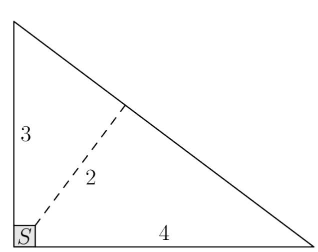
15. **(2021 AMC 12B #14)** Let $ABCD$ be a rectangle and let $\overline{DM}$ be a segment perpendicular to the plane of $ABCD$. Suppose that $\overline{DM}$ has integer length, and the lengths of $\overline{MA}, \overline{MC}$, and $\overline{MB}$ are consecutive odd positive integers (in this order). What is the volume of pyramid $MABCD$?

16. **(2022 AMC 10B # 16)** The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?

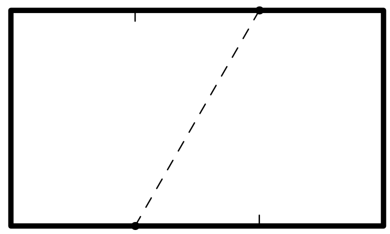


17. **(2021 AMC 10B #21)**A square piece of paper has side length 1 and vertices A, B, C, and D in that order. As shown in the figure, the paper is folded so that vertex C meets edge $\overline{AD}$ at point C', and edge $\overline{BC}$ intersects edge $\overline{AB}$ at point E. Suppose that $C'D = \frac{1}{3}$. What is the perimeter of triangle $\triangle AEC'$?

18. **(2018 AMC 10A #23)**Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



19. **(2014 AMC 10A #23)** A rectangular piece of paper whose length is $\sqrt{3}$ times the width has area A. The paper is divided into three equal sections along the opposite lengths, and then a dotted line is drawn from the first divider to the second divider on the opposite side as shown. The paper is then folded flat along this dotted line to create a new shape with area B. What is the ratio $\frac{B}{A}$?



20. **(2002 AMC 12B #20)** Let $\triangle XOY$ be a right-angled triangle with $m\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.

21. **(2015 AMC 12A # 20)** Isosceles triangles T and T' are not congruent but have the same area and the same perimeter. The sides of T have lengths 5, 5, and 8, while those of T' have lengths a, a, and b. Find b.

22. **(2001 AIME I #4)** In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.