# Planar Geometry - Perimeter, Area, and Chords

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## 1  The Isoperimetric Inequality

The *isoperimetric problem* is a classical problem to find the largest possible area of a plane figure with a specified perimeter. Similar problems, like *Dido's problem* (named for the first queen of Carthage) had been posited as early as 800 BCE, and ancient Greek geometers had intuited that the answer must be a circle. To see why, it will help to measure the area of some figures. Below are three shapes with perimeter 1 (not drawn to scale).
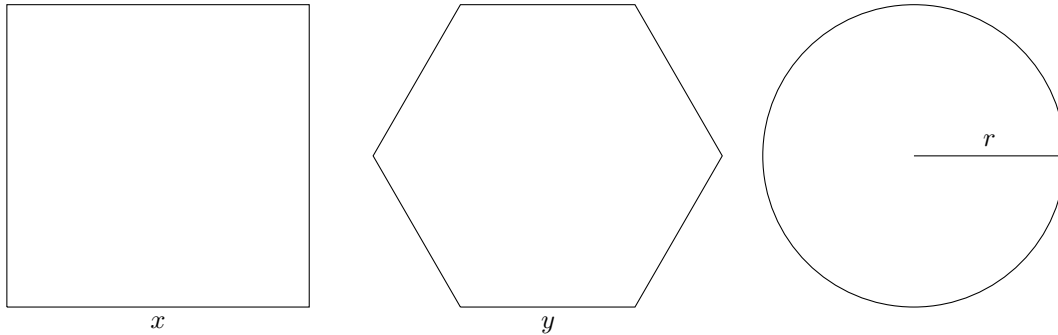


**Problem 1** *Given that the above polygons are regular (and that the shape on the right is a circle), and that all shapes have perimeter 1, find the values of $x, y$, and $r$, and then calculate the area of each shape. Which has the biggest area?*

*Solution*:  $x = 1/4, y = 1/6, r = 1/(2\pi)$, which gives the areas $1/16, \sqrt{3}/24, 1/(4\pi)$. The square has the smallest area, followed by the hexagon, followed by the circle.

Though the Greeks were the first to solve the isoperimetric problem, they did not rigorously prove that the circle is the answer - in fact, this fact was not proven until the $19^{th}$ century. We will give a proof, but first we should fix some more modern terminology. We will say that a *curve* is a shape that can be drawn in the plane without lifting the pen, and that a *region* is the inside of some curve, the length of which is the *perimeter* of the region. Another way to express that the circle maximizes the area for a given perimeter is in inequality form:

**Theorem 1** *(Isoperimetric inequality) Given a region in the plane with area $A$ and perimeter $P$,*

$$4\pi A \leq P^2$$

*with equality if and only if the region is the inside of a circle. The quantity*

$$Q := \frac{4\pi A}{P^2}$$

*is called the **isoperimetric quotient**.*

**Problem 2** *Show that the Isoperimetric Inequality is equivalent to the statement that the isoperimetric quotient $Q \leq 1$.*
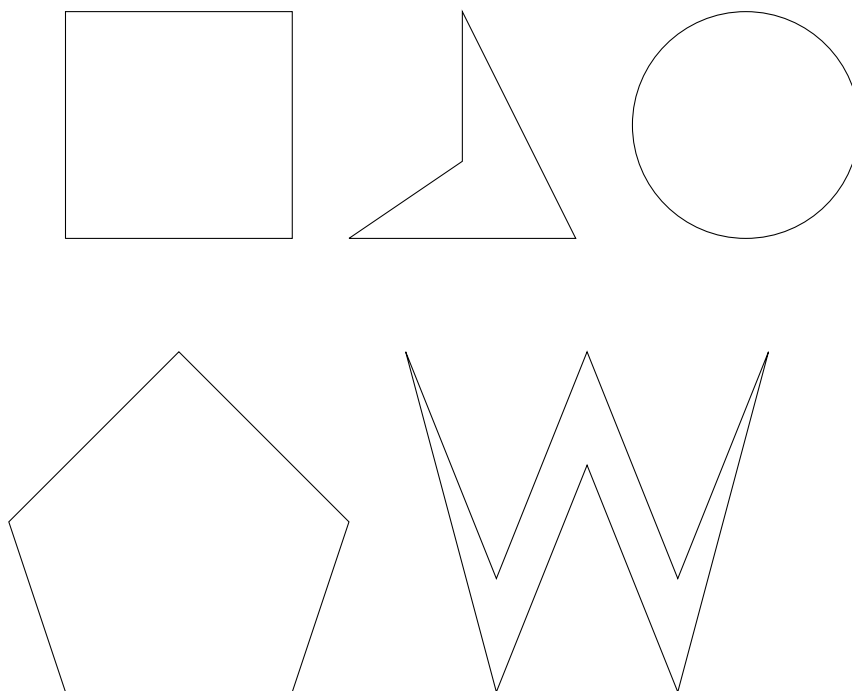
*Solution*: Divide both sides of the inequality by $P^2$, which we know is positive.

## 2 Convex Regions

As usual, let us reduce the problem to a simpler case.

**Definition 1** *A region in the plane is **convex** if it contains the line segment between any two points inside.*

**Problem 3** *Are the following regions convex? If not, disprove it by drawing an example line segment.*



*Solution*: The arrow and W are not convex, and the other shapes are.

**Problem 4** *For each region in the previous problem that wasn't convex, find another region with the same perimeter and larger area.*

*Solution*: Take the line segment drawn in the previous problem and reflect the part of the boundary that goes inside over it. For example:
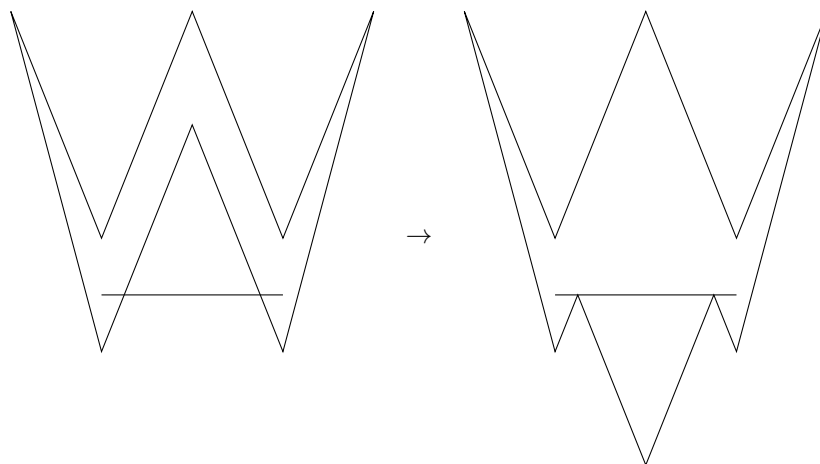


**Problem 5** *Explain why it suffices to prove Theorem 1 in the case that the region is convex.*

*Solution*: If a region is not convex, then some line segment between two points inside goes outside the region. We can reflect the portion of the perimeter between where it goes outside (just like the previous problem) to increase the area while keeping the same perimeter. Repeating this will make the region convex, while only increasing the area, so if the inequality is true for convex regions, it must also be true for our region.

**Problem 6** *(Bonus) Show that a polygon is convex if and only if all of its interior angles measure less than 180 degrees.*

**Problem 7** *One important class of examples of convex regions is given by the insides of regular polygons. Compute the isoperimetric quotients for a regular triangle and pentagon. (Bonus) Can you do a general regular n-gon as well?*

*Solution*: The $n$-gon has isoperimetric quotient

$$Q = \frac{\pi}{n \tan(\pi/n)}$$

shown by splitting it into triangles from the center.

# 3 Symmetrization and Steiner's Method

The first modern progress made towards the isoperimetric problem was by Swiss mathematician Jakob Steiner (1838). Though modern proofs (using calculus) are generally faster, Steiner's proof does not use calculus, as he disliked using analytical methods to solve geometric problems out of principle. So it will be very illustrative to go through Steiner's proof, which relies on the following symmetrization technique.

**Definition 2** *Given a convex region R and a line L in the plane, the (discrete) **Steiner symmetrization** of R about L is given by the following procedure.*

1. *Draw the perpendicular lines to L that are tangent to R. (There are two, one at each "end".)*

2. *Draw any number n of other perpendicular lines to L between the first two. These will intersect R twice.*

3. *On each line perpendicular to L, measure the the distance between the two points where it intersects R (which is zero if they are tangent).*

4. *On each line perpendicular to L, draw points above and below L such that their distance equals the distance measured in the previous step, and the midpoint between them lies on L. (For the tangent lines, draw the point on L.)*

5. *Connect these points with line segments.*

The following diagram shows an example of a Steiner symmetrization of a triangle.
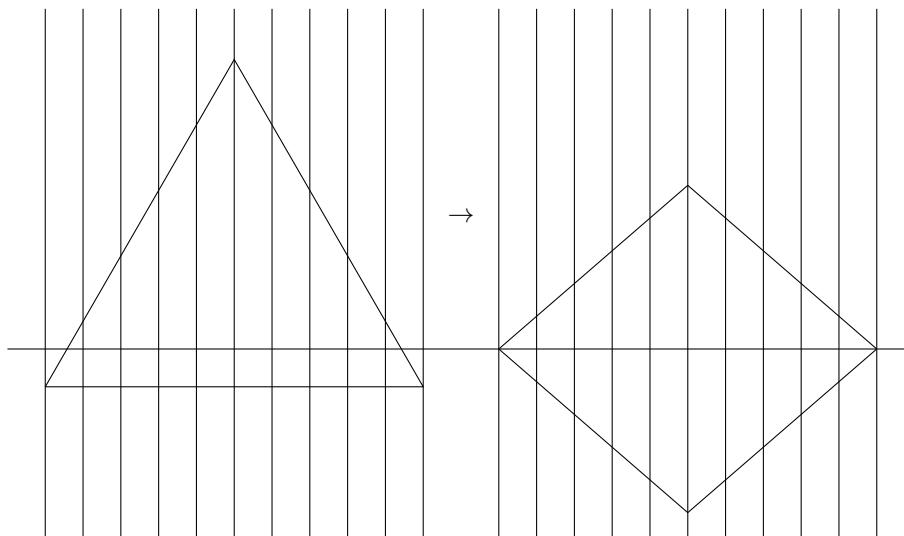


**Problem 8** *In Step 2, we know that these lines intersect R twice. Why?*

*Solution*: Since the lines go between the two ends of the region, they must intersect it. They must intersect it an even number of times since each intersection is an entrance or an exit. If one such line intersects the region more than twice, then the segment between the second and third intersections cannot be inside the region. But then that segment is a line segment between two points in the region that does not lie in the region, which is impossible since we assumed the region is convex. Therefore these lines must intersect R exactly twice.
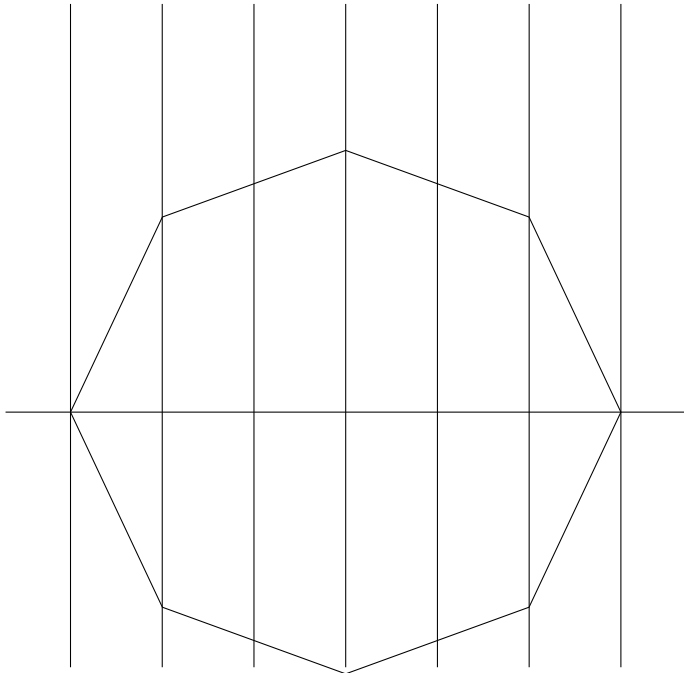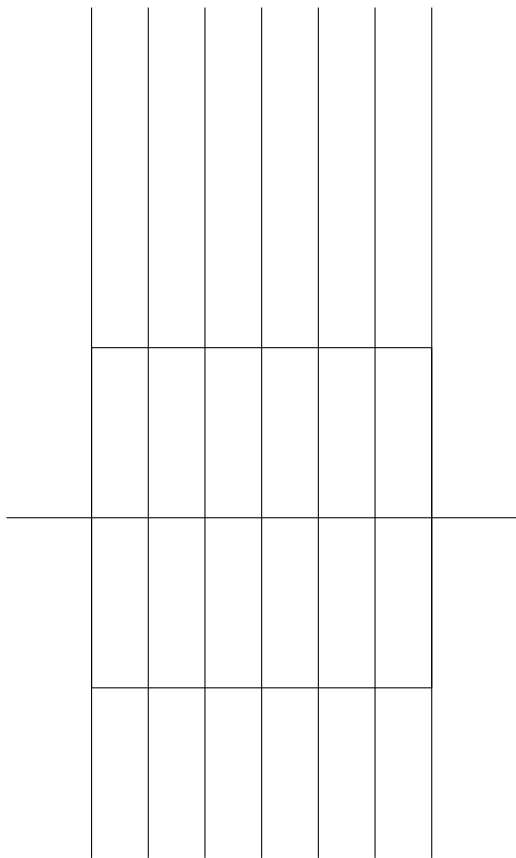
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**Problem 9** *Find the following Steiner symmetrizations (continued on the next page).*
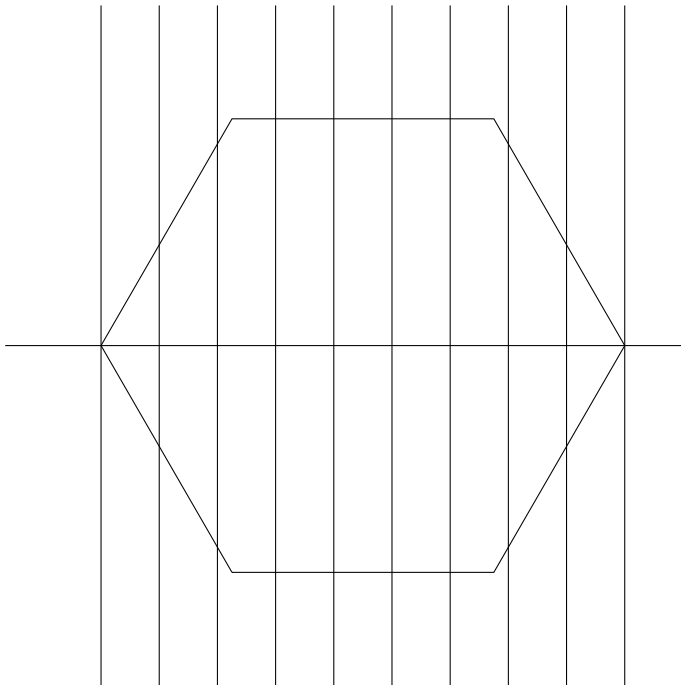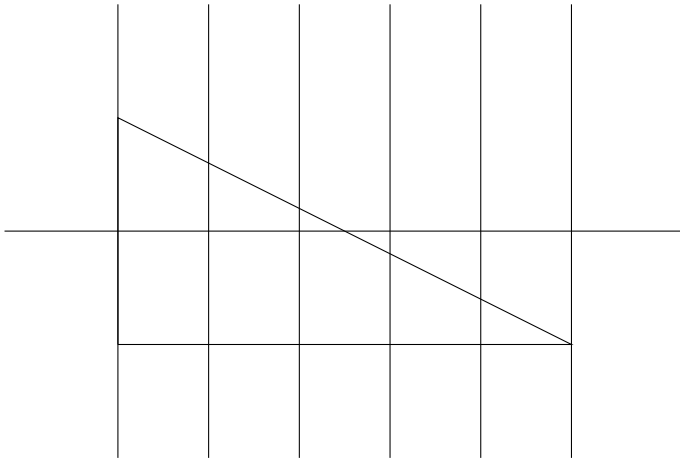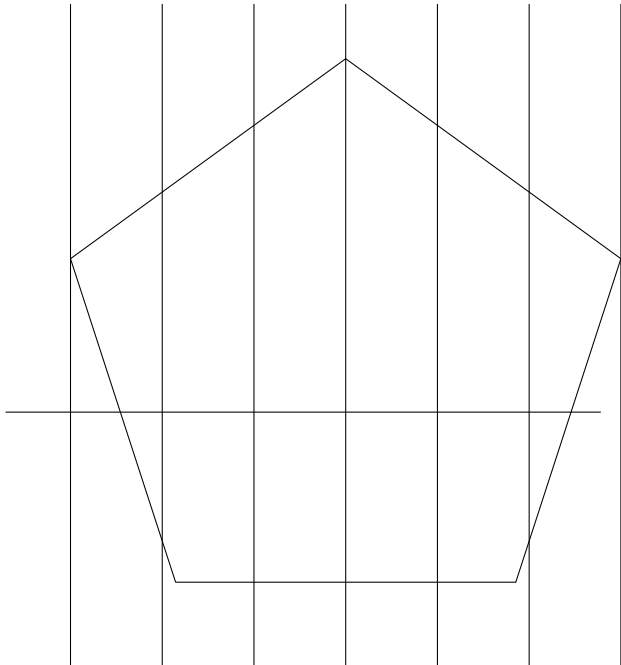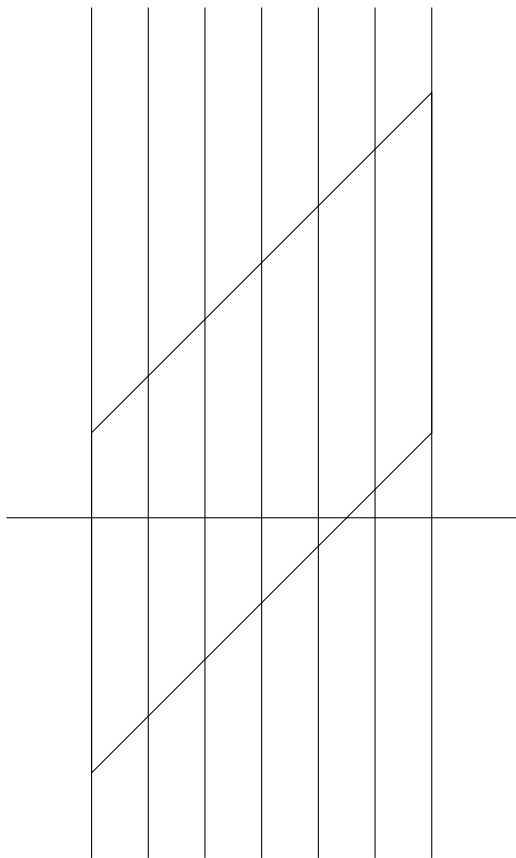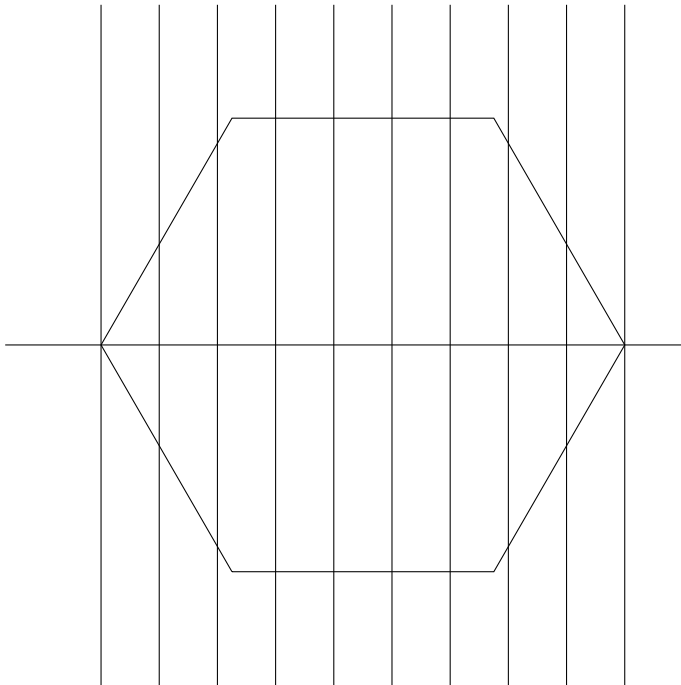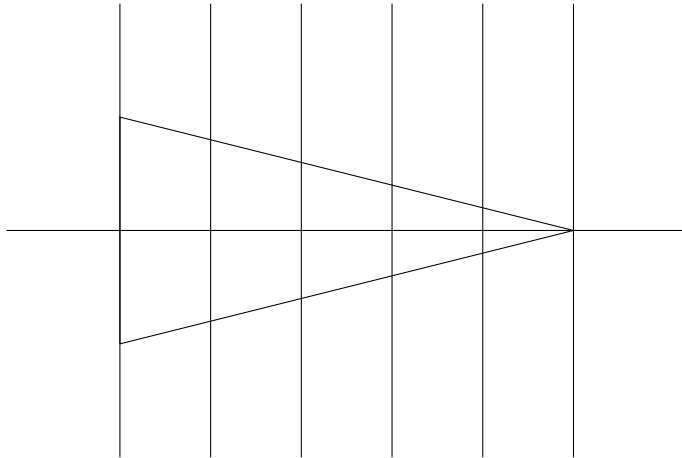
**Problem 10** *What happens when we repeat Steiner symmetrization?*

*Solution*: In order to repeat the process, we need the symmetrization of a convex region to be convex. Each trapezoid between formed between an adjacent pair of perpendicular lines is convex – this can be shown by either looking at the angles, which must be symmetric, or by considering a line segment contained in that part before symmetrization, and noting that it gets translated the same amount. Since the trapezoids have to attach at the ends by definition, the symmetrized region is convex.

When we symmetrize a polygon once (and assuming the region is not symmetric about the line, as in the second example above), we end up with a polygon with more sides, and all the same symmetries. (regular, etc.) (Make sure students convince themselves of this!) Therefore repeating the process (using different lines, since using the same line would do nothing) will yield regular $n$-gons for larger $n$, which approach a circle.

**Problem 11** *Show that Steiner symmetrization preserves the area of a region.*

*Solution*: This is just the area of a trapezoid.

**Problem 12** *Using a ruler (or your geometric reasoning), approximate the perimeters of each region from Problem 9 before and after Steiner symmetrization.*

**Problem 13** *(Bonus) Prove that Steiner symmetrization can only decrease the perimeter of a region.*

**Problem 14** *Use Steiner symmetrization to give a proof of the Isoperimetric Inequality.*

*Solution*: As seen in Problems 10, 11, and 13, applying Steiner symmetrization to a region that's not symmetric about every line (i.e. a circle) will preserve its area and decrease its perimeter. Repeatedly doing this will give a circle with the same area, which will have less perimeter, so every region that's not a circle has $Q < 1$ (and clearly, $Q = 1$ for the circle).

**Problem 15** *(Bonus) Now consider regions in 3-dimensional space. We can loosely define these as the inside of a surface.*

- *Nature is very good at optimizing geometric quantities. For instance, to minimize the surface tension on a soap bubble, it should have the smallest possible surface area for its volume. Based on your knowledge of soap bubbles, what region has the smallest possible surface area for its volume?*

- *Write down a version of Theorem 1 that holds for 3-dimensional regions.*

- *Can you prove this inequality by a similar technique?*

- *In the plane, there are no shapes with zero area and length (there are points, but they don't really add anything to a shape) However, in 3-dimensional space, there are at least some interesting shapes with zero surface area and zero volume. Describe some of them, and why they change the statement of the Isoperimetric Inequality slightly.*