

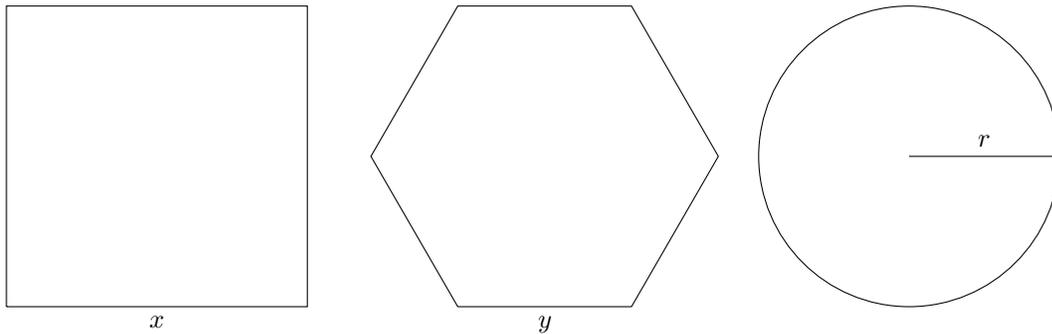
Perimeter, Area, and the Isoperimetric Inequality

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Advanced 1 - Fall 2023

1 The Isoperimetric Inequality

The *isoperimetric problem* is a classical problem to find the largest possible area of a plane figure with a specified perimeter. Similar problems, like *Dido's problem* (named for the first queen of Carthage) had been posed as early as 800 BCE, and ancient Greek geometers had intuited that the answer must be a circle. To see why, it will help to measure the area of some figures. Below are three shapes with perimeter 1 (not drawn to scale).



Problem 1 Given that the above polygons are regular (and that the shape on the right is a circle), and that all shapes have perimeter 1, find the values of x , y , and r , and then calculate the area of each shape. Which has the biggest area?

Though the Greeks were the first to solve the isoperimetric problem, they did not rigorously prove that the circle is the answer - in fact, this fact was not proven until the 19th century. We will give a proof, but first we should fix some more modern terminology. We will say that a *curve* is a shape that can be drawn in the plane without lifting the pen, and that a *region* is the inside of some curve, the length of which is the *perimeter* of the region. Another way to express that the circle maximizes the area for a given perimeter is in inequality form:

Theorem 1 (*Isoperimetric inequality*) *Given a region in the plane with area A and perimeter P ,*

$$4\pi A \leq P^2$$

with equality if and only if the region is the inside of a circle. The quantity

$$Q := \frac{4\pi A}{P^2}$$

*is called the **isoperimetric quotient**.*

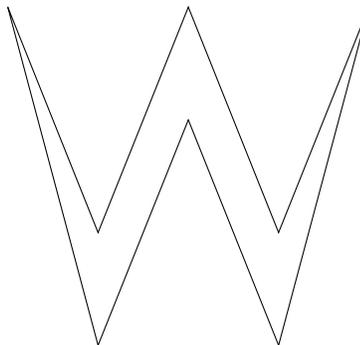
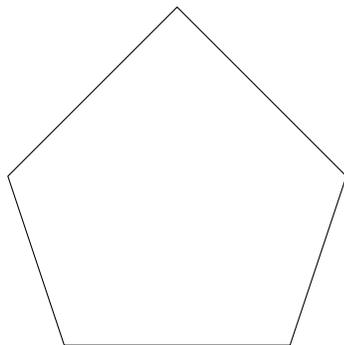
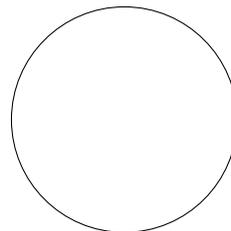
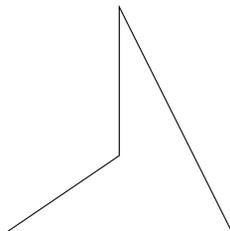
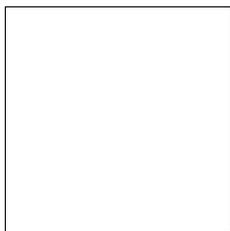
Problem 2 *Show that the Isoperimetric Inequality is equivalent to the statement that the isoperimetric quotient $Q \leq 1$.*

2 Convex Regions

As usual, let us reduce the problem to a simpler case.

Definition 1 *A region in the plane is **convex** if it contains the line segment between any two points inside the region.*

Problem 3 *Are the following regions convex? If not, disprove it by drawing an example line segment.*



Problem 4 For each region in the previous problem that wasn't convex, find another region with the same perimeter and larger area.

Problem 5 Explain why it suffices to prove Theorem 1 in the case that the region is convex.

In general, shapes that "look" convex are. The definition of a convex polygon, in particular, might have looked different in geometry class. In the following problem, we show that they are the same.

Problem 6 (Bonus) Show that a polygon is convex if and only if every interior angle measures less than 180 degrees.

Problem 7 Show that regular polygons are convex. (Hint: You can use the result of Problem 6 without proof.) Compute the isoperimetric quotients for a regular triangle and pentagon. (Bonus) Can you do a general regular n -gon as well?

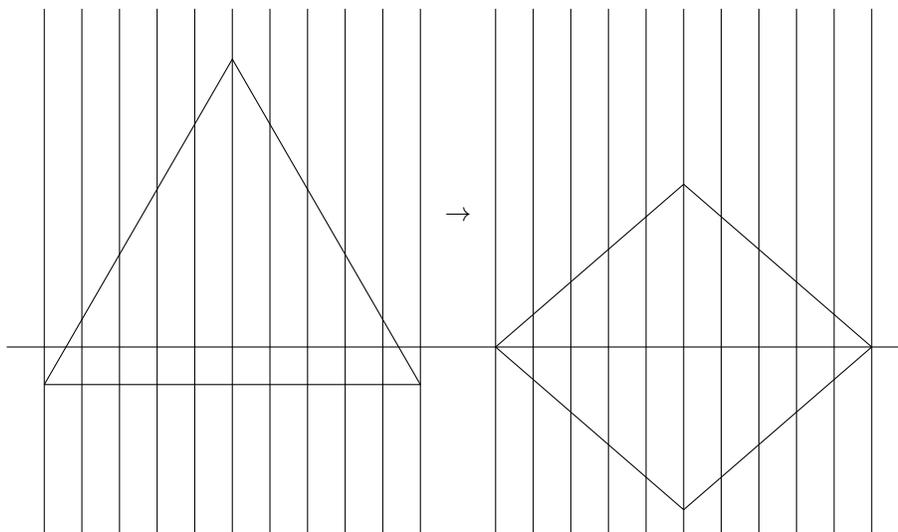
3 Symmetrization and Steiner's Method

The first modern progress made towards the isoperimetric problem was by Swiss mathematician Jakob Steiner (1838). Though modern proofs (using calculus) are generally faster, Steiner's proof does not use calculus, as he disliked using analytical methods to solve geometric problems out of principle. So it will be very illustrative to go through Steiner's proof, which relies on the following symmetrization technique.

Definition 2 Given a convex region R and a line L in the plane, the (discrete) **Steiner symmetrization** of R about L is given by the following procedure.

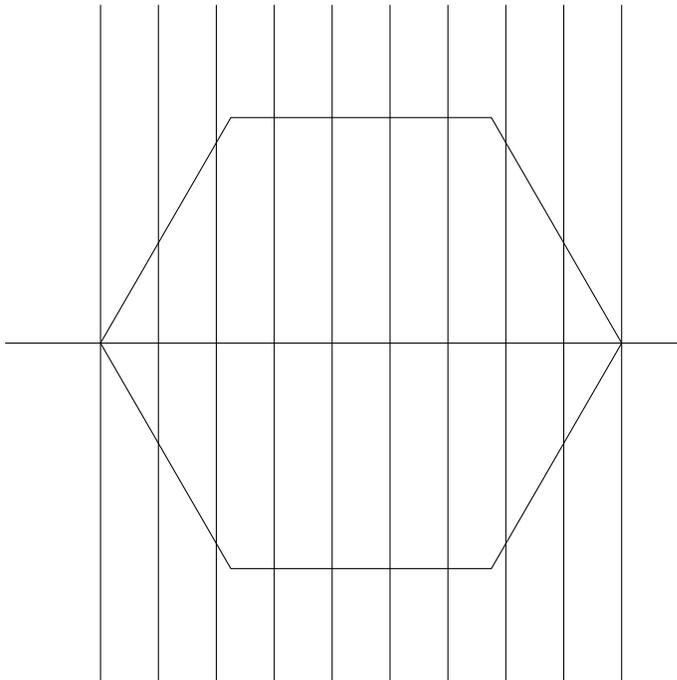
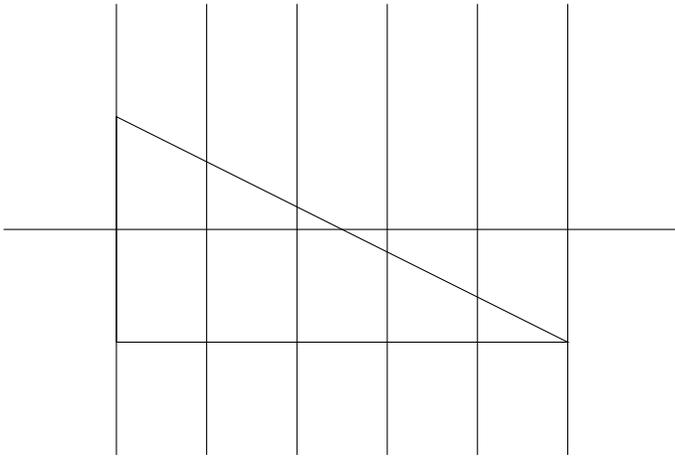
1. Draw the perpendicular lines to L that are tangent to R . (There are two, one at each "end".)
2. Draw any number n of other perpendicular lines to L between the first two. These will intersect R twice.
3. On each line perpendicular to L , measure the distance between the two points where it intersects R (which is zero if they are tangent).
4. On each line perpendicular to L , draw points above and below L such that their distance equals the distance measured in the previous step, and the midpoint between them lies on L . (For the tangent lines, draw the point on L .)
5. Connect these points with line segments.

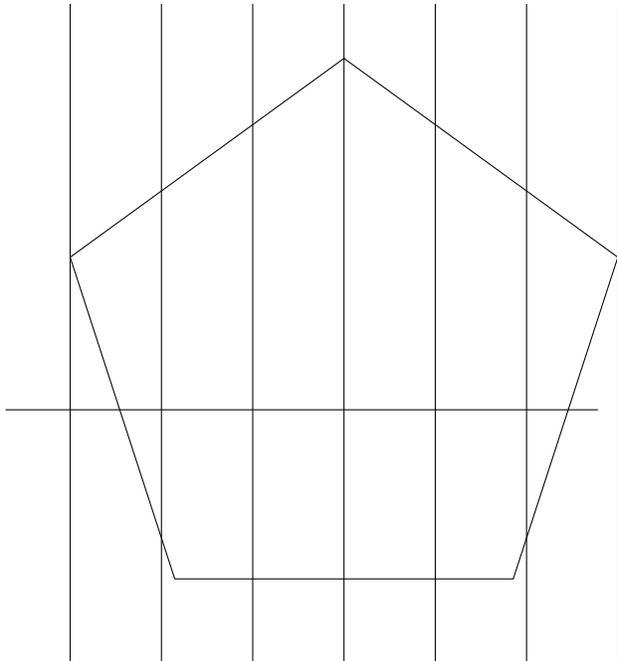
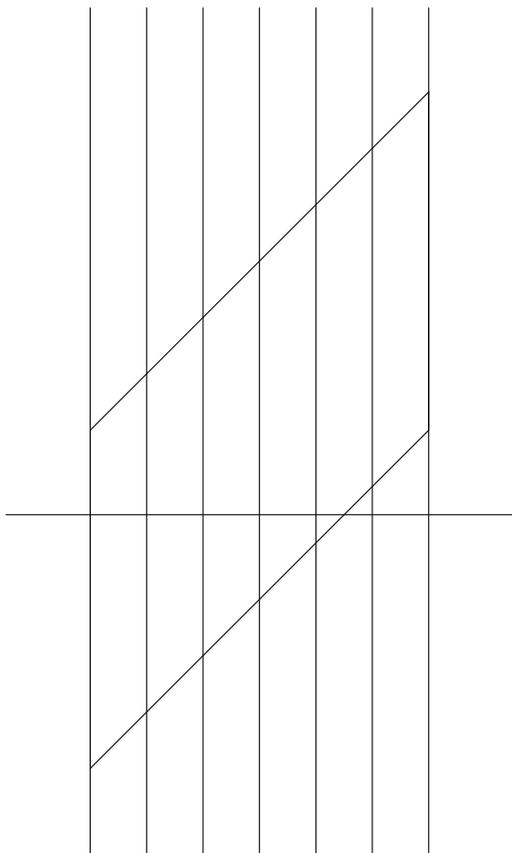
The following diagram shows an example of a Steiner symmetrization of a triangle.



Problem 8 In Step 2, we know that these lines intersect R twice. Why?

Problem 9 Find the following Steiner symmetrizations (continued on the next page).





Problem 10 *Show that we can repeatedly apply Steiner symmetrization - what happens if we do so?*

Problem 11 *Show that Steiner symmetrization preserves the area of a region.*

Problem 12 *Using a ruler (or your geometric reasoning), approximate the perimeters of each region from Problem 9 before and after Steiner symmetrization.*

Problem 13 (Bonus) Prove that Steiner symmetrization can only decrease the perimeter of a region.

Problem 14 Use Steiner symmetrization to give a proof of the Isoperimetric Inequality. (Hint: You can use the result of Problem 13 without proof.)

Problem 15 (Bonus) Now consider regions in 3-dimensional space. We can loosely define these as the inside of a surface.

- Nature is very good at optimizing geometric quantities. For instance, to minimize the surface tension on a soap bubble, it should have the smallest possible surface area for its volume. Based on your knowledge of soap bubbles, what region has the smallest possible surface area for its volume?
- Write down a version of Theorem 1 that holds for 3-dimensional regions.
- Can you prove this inequality by a similar technique?
- In the plane, there are no shapes with zero area and length. (There are points, but they don't really add anything to a shape) However, in 3-dimensional space, there are at least some interesting shapes with zero surface area and zero volume. Describe some of them, and why they change the statement of the Isoperimetric Inequality slightly.