Olympiads Week 5: Polynomials 2

ORMC

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Remember to *write down your solutions, as proofs.* You don't have to start by writing out a full proof to every problem you try, but once you've solved a problem or two, take a few minutes to write out a proof as if this was being graded at an Olympiad.

Problem 0.1 (Putnam 2005 B1)**.** Find a nonzero polynomial $P(x, y)$ such that for all $a \in \mathbb{R}$, $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$.

($\lfloor x \rfloor$ is the greatest integer with $\lfloor x \rfloor \leq x$.)

1 Viète's Formulas

Non-competition problems in this section are from Putnam and Beyond.

Theorem 1.1 (Viète's Formulas)**.** *Let $P(x) = a_n x^n + \cdots + a_0$ be a polynomial with coefficients in $\mathbb{C}$. Then the fundamental theorem of algebra tells us that we can factor it as*

$$P(x) = a_n(x - x_1)\dots(x - x_n)$$

for some complex roots $x_1, \dots, x_n$.

Viète's Formulas *say that*

$$(-1)^k \frac{a_{n-k}}{a_n} = S_k(x_1, \dots, x_n),$$

where $S_k(x_1, \dots, x_n)$ is the elementary symmetric polynomial of degree k*, meaning it is the sum of all distinct products of k distinct variables in $x_1, \dots, x_n$.*

Most famously, in the cases $k = 1, k = n$, we get

$$-\frac{a_{n-1}}{a_n} = x_1 + \cdots + x_n$$

and

$$(-1)^n \frac{a_0}{a_n} = x_1 \dots x_n.$$

Problem 1.2 (AIME II 2008 Problem 7)**.** Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

Problem 1.3. Find the zeros of the polynomial $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$ knowing that the sum of two of them is 4.

Problem 1.4. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}, a_{n-2} = \binom{n}{2}$, and that all roots are real, find the remaining coefficients.

Problem 1.5. Let a, b, c be real numbers. Show that $a \geq 0, b \geq 0$, and $c \geq 0$ if and only if $a + b + c \geq 0$, $ab + bc + ca \geq 0$, and $abc \geq 0$.

Problem 1.6 (BAMO 2022 Problem 5)**.** Sofiya and Marquis are playing a game. Sofiya announces to Marquis that she's thinking of a poly- nomial of the form $f(x) = x^3 + px + q$ with three integer roots that are not necessarily distinct. She also explains that all of the integer roots have absolute value less than (and not equal to) N, where N is some fixed number which she tells Marquis. As a "move" in this game, Marquis can ask Sofiya about any number x and Sofiya will tell him whether $f(x)$ is positive, negative, or zero. Marquis's goal is to figure out Sofiya's polynomial. If $N = 3 \cdot 2^k$ for some positive integer k, prove that there is a strategy which allows Marquis to identify the polynomial after making at most $2k + 1$ "moves".

2 Irreducible Polynomials

Non-competition problems in this section are from Sucharit Sarkar's Math 100.

Problem 2.1. Show that $x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$, where a, b, c, d are positive integers, is divisible by $x^3 + x^2 + x + 1$.

Hint: $x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$.

If $f(x)$ is an integer polynomial, we say that it's *irreducible* when there are no nonconstant polynomials $g(x)$ and $h(x)$ with $f(x) = g(x)h(x)$.

Here are two results that are useful in factoring polynomials with integer coefficients into irreducibles.

Theorem 2.2 (Rational-Root Theorem)**.** *If $P(x) = a_n x^n + \cdots + a_0$ is a polynomial with integer coefficients, and if the rational number $\frac{r}{s}$ (r and s are relatively prime) is a root of $P(x) = 0$, then r divides a_0 and s divides a_n.*

Lemma 2.3 (Gauss's Lemma)**.** *Let $P(x)$ be a polynomial with integer coefficients. If $P(x)$ can be factored into a product of two polynomials with rational coefficients, then $P(x)$ can be factored into a product of two polynomials with integer coefficients.*

Problem 2.4. Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial of degree n with integral coefficients. If a_0, a_n and $f(1)$ are odd, prove that $f(x) = 0$ has no rational roots.

Problem 2.5. For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$?

2.1 Eisenstein's Criterion

As a hint for an IMO problem, let's prove a tool called *Eisenstein's Criterion* that helps determine when integer polynomials are irreducible.

Problem 2.6. Prove *Eisenstein's Criterion*:

Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be an integer polynomial, and let p be a prime such that

- For $i < n$, $p | a_i$
- $p \not| a_n$
- $p^2 \not| a_0$.

Then $f(x)$ is irreducible.

Problem 2.7 (IMO 1993 Problem 1)**.** Let $n > 1$ be an integer. Prove that there are no nonconstant polynomials $g(x)$ and $h(x)$ with integer coefficients such that

$$g(x)h(x) = x^n + 5x^{n-1} + 3.$$

3 Competition Problems

Problem 3.1 (HMMT 2007)**.** The complex numbers $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

Problem 3.2 (BAMO 2012 Problem 7)**.** Find all nonzero polynomials $P(x)$ with integer coefficients that satisfy the following property: whenever a and b are relatively prime integers, then $P(a)$ and $P(b)$ are relatively prime as well. Prove that your answer is correct. (Two integers are relatively prime if they have no common prime factors. For example, -70 and 99 are relatively prime, while -70 and 15 are not relatively prime.)

Problem 3.3 (BAMO 2017 Problem 5)**.** Call a number T persistent if the following holds: Whenever a, b, c, d are real numbers different from 0 and 1 such that

$$a + b + c + d = T$$

and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = T,$$

we also have

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} + \frac{1}{1-d} = T.$$

What numbers are persistent?

Problem 3.4 (USAMO 1995 Problem 4)**.** Suppose $q_0, q_1, q_2, \ldots$ is an infinite sequence of integers satisfying the following two conditions:

- $m - n$ divides $q_m - q_n$ for $m > n \geq 0$,
- there is a polynomial P such that $|q_n| < P(n)$ for all n.

Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.

Problem 3.5 (Putnam 2019 Problem B5)**.** Let F_m be the mth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.