# More on Counting (Solutions)

## 1   Counting Review

**Problem 1** *Compute the following:*

$a.$   $1! = 1$                      $b.$   $2! = 2$

$c.$   $4! = 24$                     $d.$   $5! = 120$

$e.$   $\dfrac{5!}{2! \cdot 3!} = 10$

$f.$   $\dfrac{1000!}{998! \cdot 2!} = 499500$

**Problem 2** *How many ways are there to place the letters 'M', 'A', 'T', 'H' together in a row, ignoring whether it's a valid word?*

$4! = 24.$

Recall that a way to choose $k$ objects out of $n$ so that the **order of the chosen objects matters** is called a **permutation**. the number of permutations is given by the following formula.

$$P(n, k) = \frac{n!}{(n - k)!} \tag{1}$$

**Problem 3** *Compute the following:*

a. $P(3, 0) = 1$ b. $P(3, 1) = 3$

c. $P(3, 2) = 6$ d. $P(3, 3) = 6$

e. $P(5, 4) = 120$

f. $P(100, 1) = 100$

**Problem 4** *There are 25 students in a class. The teacher wants to choose one student from the class to erase the board, and another student to clean the floor. In how many ways can the teacher choose the students?*

$25 \times 24 = 600.$

Suppose now the order of the chosen things does not matter. A way to choose $k$ objects out of $n$ so that the order of the chosen objects **does not matter** is called a **<u>combination</u>**. The formular is given by

$$\binom{n}{k} = \frac{n!}{(n-k)!\, k!} = C(n, k) \tag{2}$$

**Problem 5** *Compute the following:*

a. $\binom{4}{0} = 1$

b. $\binom{4}{1} = 4$

c. $\binom{4}{2} = 6$

d. $\binom{4}{3} = 4$

e. $\binom{4}{4} = 1$

f. $\binom{100}{1} = 100$

**Problem 6** *There are 25 students in a class. The teacher wants to choose 2 students from the class to erase the board. In how many ways can the teacher choose the students?*

$\binom{25}{2} = 300$

*What is the difference between Problem 4 and 6?*

*Order matters in Problem 4 but not in Problem 6.*

## 2 Let's Flip Coins

Form groups of 4 or 5 people. Toss 4 coins one time each and record how many heads you get in the space below. Repeat this 20 times, and make sure you write down the number of heads each time.

| Trial # | The Number of Heads | Trial # | The Number of Heads |
|---------|---------------------|---------|---------------------|
| 1 | | 11 | |
| 2 | | 12 | |
| 3 | | 13 | |
| 4 | | 14 | |
| 5 | | 15 | |
| 6 | | 16 | |
| 7 | | 17 | |
| 8 | | 18 | |
| 9 | | 19 | |
| 10 | | 20 | |

**Problem 7** *Out of the 20 trials you did, what is the percentage of times that you got the following outcomes?*

*0 heads:*

*1 head:*

*2 heads:*

*3 heads:*

*4 heads:*

**Problem 8** *We toss 4 coins at the same time. List all possible ways to get the following outcomes. Use H to represent heads and T to represent tail*

0 heads: TTTT.

1 head: HTTT, THTT, TTHT, TTTH

2 heads: HHTT, HTHT, HTTH, THHT
THTH, TTHH

3 heads: HHHT, HHTH, HTHH, THHH

4 heads: HHHH

**Problem 9** *What is the total number of outcomes you can get by tossing 4 coins. (Order matters!!) Calculate this using math, then verify it by counting all possible outcomes in the previous problem.*

$2^4 = 16$.

**Problem 10** *When tossing 4 coins, what is the probability of the following outcomes? Calculate using combination formula, then verify by counting the results in Problem 8.*

*0 heads:* $\dfrac{\binom{0}{4}}{16} = \dfrac{1}{16}$

*1 head:* $\dfrac{\binom{1}{4}}{16} = \dfrac{1}{4}$

*2 heads:* $\dfrac{\binom{2}{4}}{16} = \dfrac{3}{8}$

*3 heads:* $\dfrac{\binom{3}{4}}{16} = \dfrac{1}{4}$

*4 heads:* $\dfrac{\binom{4}{4}}{16} = \dfrac{1}{16}$

*Compare your answer to Problem 7 and 10. Is the result from your experiment in Problem 7 the same or different from your computation in Problem 10?*

**Problem 11** *You toss a fair coin 10 times. What is the chance that you get 5 tails?*

$$\frac{\binom{10}{5}}{2^{10}} = \frac{252}{1024}$$

**Problem 12** *You toss a fair coin 16 times. (Hint: $2^{16} = 65536$)*

- *What is the chance that you get no tails?*

$$\frac{1}{65536}$$

- *What is the chance that you get strictly less than 3 tails?*

$$\frac{\binom{16}{0} + \binom{16}{1} + \binom{16}{2}}{65536} = \frac{137}{65536}$$

- *What is the chance that you get at least 3 tails?*

$$1 - \frac{137}{65536}$$

# 3   More Probability Problems

**Problem 13** *A club has 10 members, 5 boys and 5 girls. Two of the members are chosen at random. What is the probability that they are both girls?*

$$\frac{\binom{5}{2}}{\binom{10}{2}} = \frac{10}{45} = \frac{2}{9}.$$

**Problem 14** *Another club has 20 members, 12 boys and 8 girls. Two of the members are chosen at random. What is the probability that a boy and a girl are chosen?*

$$\frac{12 \times 8}{\binom{20}{2}} = \frac{96}{190} = \frac{48}{95}.$$

**Problem 15** *What is the probability that a random arrangement of the letters in the word 'SEVEN' will have both E's next to each other?*

This problem is the same as the following: there are 5 boxes in a row, and we choose 2. What is the probability that the 2 boxes we chose are next to each other. The answer is

$$\frac{4}{\binom{5}{2}} = \frac{2}{5}.$$

**Problem 16** *2 vertices of an octagon are chosen at random. What is the probability that they are adjacent?*

$$\frac{8}{\binom{8}{2}} = \frac{2}{7}$$

**Problem 17** *A fair coin is tossed 4 times. What is the chance of getting more HEADS than TAILS?*

Just look at Problem 10. Getting more heads than tails means getting 3 or 4 heads, so the answer is

$$\frac{1}{4} + \frac{1}{16} = \frac{5}{16}.$$