NUMBER BASES

MATH CIRCLE (ADVANCED) 10/21/2012

We write the number three thousand six hundred fifty seven as

$$3657 = 3 \cdot 1000 + 6 \cdot 100 + 5 \cdot 10 + 7 = 3 \cdot 10^3 + 6 \cdot 10^2 + 5 \cdot 10^1 + 7 \cdot 10^0.$$

Is there anything special about using 10? From a mathematical point of view, the answer is no.

Proposition: Fix some n (called the base). We can write any number uniquely in the form

$$a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n + a_0 n^0$$

where each a_i takes values from 0 to $n - 1$.

To avoid ambiguity, we will write the base as a subscript. For example, in base 6, the number fifty (50_{10}) is $1 \cdot 6^2 + 2 \cdot 6 + 2 \cdot 6^0 = 122_6$.

1) Convert the following numbers to base 10.

a) $1232_4 = 110_{10}$

b) $10120_3 = 96_3$

c) $723_9 = 588_{10}$

d) $100111_2 = 39_{10}$

e) $1232_5 = 192_{10}$

2) Convert the number 98_{10} into: (can you come up with a fast way to do this?)

a) base 7: 200_7

b) base 4: 1202_4

c) base 3: 10122_3

d) base 2: 1100010_2

3) Write out multiplication tables for base 2 and base 3.

Base 2	0	1
0	0	1
1	1	1

Base 3	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

We can add/multiply numbers using the same methods we know for base 10.

4) Calculate (in the same base as given):

a) $11010_2 + 10011_2 = 101101_2$

b) $21012_3 + 121201_3 = 212220_3$

c) $1001_2 \cdot 101_2 = 101101_2$

d) $2111_3 \cdot 122_3 = 1120012_3$

5) Is it possible that the following statements are true in some number base system? Is so, what base?

a) $3 \cdot 4 = 10$: base 12

b) both $3 + 4 = 10$ and $3 \cdot 4 = 15$: base 7.

c) both $2 + 3 = 5$ and $2 \cdot 3 = 11$:

Not possible. First equation needs base ≥ 6, while second needs base 5.

6) State and prove a condition (involving the representation of a number) which allows us to determine whether the number is even or odd:

a) in the base 3 system.

b) in the base n system.

If n is even: a number is odd if and only if the last digit is odd in the representation in base n.

If n is odd: a number is odd if and only if there are an odd number of odd digits in the representation in base n.

7) An evil king wrote three secret two-digit numbers a, b, c. A handsome prince must name three numbers X, Y, Z, after which the king will tell him the sum $aX + bY + cZ$. The prince must then name all three of the King's numbers, or he will be executed. Help out the prince!

Check that $X = 100^2, Y = 100, Z = 1$ works.

8)* Prove the proposition stated at the beginning of the handout.

9)* Prove that from the set $0, 1, 2, \ldots 3^k - 1$ one can choose 2^k numbers so that none of them can be represented as the arithmetic mean of some pair of the chosen numbers.

Hint: Work in base 3. The 2^k numbers that we want are those that only have 0 and 1 in their base 3 representation.

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"