Combinatorics

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1 Warm Up

The sum of two prime numbers is 85. What is the product of these two prime numbers?

(A) 85 **(B)** 91 **(C)** 115 **(D)** 133 **(E)** 166

How many positive integer factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1, 2, 3, 4, 6$, and 12.)

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

What is the ratio of the least common multiple of 180 and 594 to the greatest common factor of 180 and 594?

(A) 110 **(B)** 165 **(C)** 330 **(D)** 625 **(E)** 660

2 Permutations and Combinations

2.1 Factorials

The factorial function (symbol: **!**) says to multiply all whole numbers from our chosen number down to 1. You should know your factorials up to 6! at least.

n	Factorial of a Number n!	Expansion	Value
1	1!	1	1
2	2!	2 × 1	2
3	3!	3 × 2 × 1	6
4	4!	4 × 3 × 2 × 1	24
5	5!	5 × 4 × 3 × 2 × 1	120
6	6!	6 × 5 × 4 × 3 × 2 × 1	720

If necessary, you can calculate the next factorials by multiplying any additional numbers by 6! The only exception to the factorial rule is 0!, so just remember that $0! = 1$.

Factorials can be used to find the number of ways in which objects in a set can be organized, which we will see below.

2.2 Permutations

A permutation is a possible arrangement of objects in a set where the order of objects matter. The number of ways to re-order all n objects is simply $n! = 1 \times 2 \cdots \times n$. To re-order r objects of of the set of n objects (when order matters), we can use a permutation:

$$\begin{aligned} {}_nP_r &= n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

This permutation formula often works best when all the objects in the set are different. If you have identical objects, you can switch their places and still have the same permutation. In this case, we need to make sure we do not over count by dividing by $n!$ for each repeated object.
If the i-th element has a_i copies, then the total number of permutations is:

$$\frac{{}_nP_r}{x_1!x_2!},$$

where x_1 and x_2 are the number of times an object in the set is repeated.

Examples:

1. Using only the letters given in each name, find the number of unique permutations of letters you can make.

   a. FATIMA

   b. ROHAN

   c. AURORA

   d. MATTHEW

2. Let S be the set of permutations of the sequence $1, 2, 3, 4, 5$ for which the first term is not 1. A permutation is chosen randomly from S. Find the probability that the second term is 2.

2.3 Combinations

A combination is a way to choose a subset from a set of elements where order does not matter. The number of ways to choose r out of n objects where order doesn't matter is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

We can also find this using Pascal's Triangle to iteratively find the *binomial coefficient*.

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          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5  10  10   5   1
1   6  15  20  15   6   1
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3 Other Counting Strategies

3.1 Complementary Counting

In some cases, it's easier to count the elements that aren't needed in a set. It would be easier to count the number of elements and subtract those that you don't need. This is called *complementary counting*.

3.2 Stars and Bars

Stars and Bars is a strategy used to sort identical elements into distinct groups. It utilizes stars ($*$) and bars ($|$).

If we had 7 people who like chocolate ice cream and 3 people who have vanilla ice cream, we can sort them as such: $*******|***$

The divider bar $|$ represents the separation between the two groups, while the stars $*$ represent the elements in each group.

In general, the number of ways to split n identical objects split among m people is equivalent to chosing $m-1$ dividers among $n+(m-1)$ positions for a total of

$$\binom{n+m-1}{m-1}.$$

4 Practice

If you have Anish, Miley, and Vera and you need to choose two of them to be the president and vice president of the math club, how many ways are there to do this?

How many 3-digit odd numbers are there with distinct digits?

A special type of license plate includes 3 distinct letters at the beginning and 4 single digit numbers after. How many such license plates exist?

The Taylor A. Swift Middle School chess team consists of two boys and three girls. A photographer wants to take a picture of the team to appear in the local newspaper. She decides to have them sit in a row with a boy at each end and the three girls in the middle. How many such arrangements are possible?

Professor Nikita has nine different language books lined up on a bookshelf: two English, three Russian, and four Spanish. How many ways are there to arrange the nine books on the shelf keeping the English books together and keeping the Spanish books together?

Minghan refuses to sit next to either William or Alexis. Dylan refuses to sit next to Ian. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

How many ways are there to choose 4 balls from a bag which contains 9 balls of different colors?

Ms. Rainelle is organizing a potluck and wants to serve 8 different dishes. Each family can bring 3 dishes. How many families can Ms. Rainelle invite so that no two families bring the same combination of dishes?

There are 9 different flavors of ice cream at an ice cream shop. Sunny wants to buy 2 or 3 scoops of different ice cream flavors. He doesn't like 2 of the flavors, mango and chocolate, together so won't buy them together . How many ways can he buy ice cream?

Kylar's Hamburger Haven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties and any collection of condiments. How many different kinds of hamburgers can be ordered?

How many distinguishable rearrangements of the letters in CONTEST have both the vowels first? (For instance, OETCNST is one such arrangement but OTETSNC is not.)

How many distinguishable arrangements are there of 1 brown tile, 1 purple tile, 2 green tiles, and 3 yellow tiles in a row from left to right? (Tiles of the same color are indistinguishable.)

Phoebe, Kylar, Connor, Sahana, Cole are sitting around a round table. Phoebe refuses to sit next to Kylar. How many different ways can they sit around the table? (Rotations of the same arrangement are not considered different).

Mika has a collection of 4 marbles: an Aggie, a Bumblebee, a Steelie, and a Tiger. She wants to display them in a row on a shelf, but does not want to put the Steelie and the Tiger next to one another. In how many ways can she do this?

William is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

Charlotte has 24 apples. In how many ways can she share them with Jakub and Tammo so that each of the three people has at least two apples?

Lauren has three friends who call her regularly. One calls her every three days, one calls her every four days, and one calls her every five days. All three called her on December 31, 2016. On how many days during the next year did she not receive a phone call from any of her friends?

There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?