Intermediate 1 **Oct 22, 2023**

Logic!

1 Knights & Knaves (& Knormals)

You go to visit a far off island of Rayland. Everyone on Rayland (other than you) is either a knight or a knave (we'll talk about knormals later). Knights will always tell the truth, while knaves will always lie.

Problem 1.1. *You come across three folks sitting in a garden. Let's call them A, B and C. You ask A if they are a knight. A answers, but you can't quite make out what they said. You ask B "What did A say?" B replies "A said that she was a knave." At this moment C says "Don't believe B! He is lying!"*

The question is, what are B and C?

Problem 1.2. *You walk on a bit further, and you see two little kids, A and B, climbing a tree. You ask them who they are and A responds "At least one of us is a knave"*

What are A and B?

Problem 1.3. *Suppose instead that A had said "Either I am a knave or B is a knight."*

What are A and B?

Problem 1.4. *Suppose instead that A had said "Either I am a knave or else two plus two equals fish."*

What could you conclude?

Problem 1.5. *You continue walking across the Rayland, and come across another three people, A, B and C, each of whom is either a knight or a knave. A and B make the following statements.*

A: All of us are knaves
B: Exactly one of us is a knight

What are A, B and C?

Problem 1.6. *Suppose instead that A and B had said the following*

A: All of us are knaves.
B: Exactly one of is is a knave

What can you conclude?

Problem 1.7. *Suppose that A had said "I am a knave, but B isn't"*

What are A and B?

Problem 1.8. *We again have three inhabitants, A, B and C, each of whom is either a knight or a knave. Two people are said to be of the same type if they are both knights or both knaves. A and B make the following statements:*

A: B is a knave.
B: A and B are of the same type.

What can you conclude?

Problem 1.9. *Again three people A, B and C. A says, "B and C are of the same type." Someone then asks C, "Are A and B of the same type?" What does C answer?*

Problem 1.10. *You walk a bit further still and see two inhabitants of Rayland laying under a tree. You ask one of them, "Are either of you a knight?" She responded, and you knew the answer to your question.*

Was the person that you asked the question to a knight or a knave? How about the other person?

Problem 1.11. *On the way back to your hotel, you lose your map. You come across a fork in the road, with an inhabitant of Rayland next to each of the two paths. Both of the inhabitants say, "I am a knight, but the other is a knave." If you can only ask a single question to either one (but not both) of them, how can you find your way back?*

You decide that you have seen quite enough of Rayland, and decide to leave to Smullville. Smullville is also inhabited by knights and knaves, but it also has a third kind of inhabitant, knormals. Knormals sometimes tell the truth, and other times lie. It is here where our next few puzzles take place

Problem 1.12. *We are given three people, A, B and C, one of whom is a knight, one of whom is a knave, and the last of whom is a knormal. They make the following statements:*

A: I am the knromal.
B: That isn't true!
C: I am not knormal.

What can you conclude?

Problem 1.13. *Two people, A and B say the following:*

A: B is a knight.
B: A is not a knight.

Prove that at least one of them is telling the truth, but is not a knight.

Problem 1.14. *This time A and B say the following:*

A: B is a knight.
B: A is a knave.

What can you conclude?

Problem 1.15. *On the island of Mullville, there is a caste system. knights belong to the highest caste, normals below, and knaves the lowest caste. Two inhabitants make the following statements:*

A: I am a lower rank than B.
B: That's not true!

What can you conclude?

Problem 1.16. *Given three people A, B and C, one of whom is a knight, one a knave, and the last a knormal. A and B say:*

A: B is of higher rank than C.
B: C is of higher rank than A.

Then C is asked, "Who has the higher rank, A or B?" What does C answer?

The last stop on our journey is the island of Merrillia. On the island of Merrillia, a knight can only marry a knave, a knave could only marry a knight, and so a knormal can only marry a knromal.

Problem 1.17. *The first people that we meet on the island are Mr. and Mrs. A. They make the following statements.*

Mr. A: My wife is not a knormal.
Mrs. A: My husband is not knormal.

What are Mr. and Mrs. A?

Problem 1.18. *Suppose instead that they had said:*

Mr. A: My wife is a knormal.
Mrs. A: My husband is knormal.

Would the answer have been different?

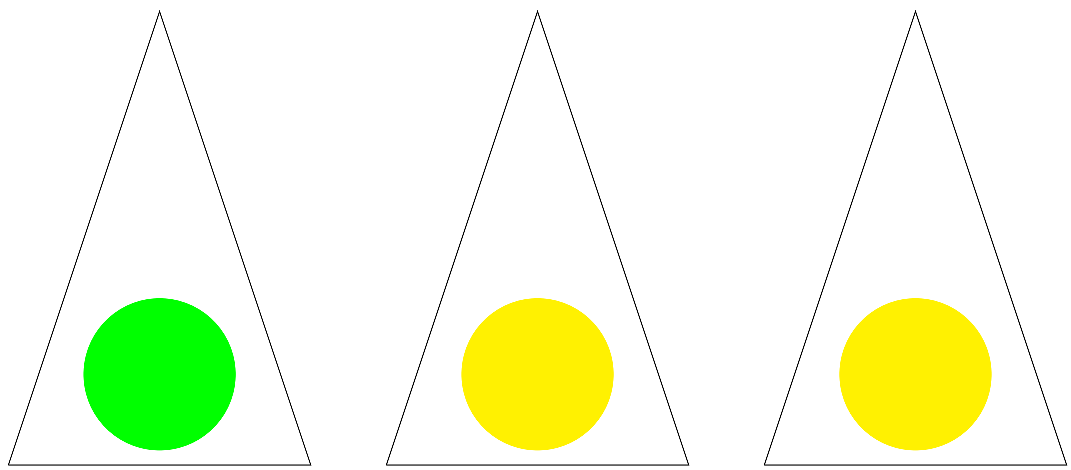
Problem 1.19. *The final people that you meet on this island are a pair of couples, Mr and Mrs A, and Mr and Mrs B. They say the following:*

Mr. A: Mr. B is a knight.
Mrs. A: That's right, Mr. B is a knight.
Mrs. B: Indeed! My husband is a knight.

What can you conclude?

2 Hat Puzzles

A part of wizardry training is developing logic skills. In the following exercises, two students will be presented with three wizard's hats, one with a green label and two with yellow labels.



The students will be asked to close their eyes. One hat will be hidden, the others will be put on the students' heads. Then the students will be asked to open their eyes and to figure out the color of their hat labels by observing that of the other student.

A student is not allowed to look at her/his own hat, or to communicate with other people. This kind of cheating will be punished by turning the cheat into a toad!

Problem 2.1. *The students, Alice and Bob, are given two hats. Alice gets the green-labeled hat, Bob gets a yellow-labeled one. Which of the students will be able to figure out the color of her/his hat? How is she/he going to do that?*

Problem 2.2. *This time, Alice and Bob are both given the yellow-labeled hats. Is there a way for them to figure out the color of their hats' labels? How can they do it?*

As the training progresses, three students will be chosen and presented with five hats, three with yellow dots and two with green. Once again, the students will be asked to close their eyes. Two hats will be hidden, three will be put on the students' heads. The students will be asked to open their eyes and to figure out the color of their hat labels by observing those of the other students.

Problem 2.3. *The students, Alice, Bob, and Charlie, are given the hats. Alice gets a yellow-labeled hat, Bob and Charlie get the hats with green labels. Which of the students will be able to figure out the color of her/his hat? How is she/he going to do that?*

Problem 2.4. *This time, Alice and Bob get yellow-labeled hats while Charlie gets the hat with a green label. Which of the students will be able to figure out the color of her/his hat? How is she/he going to do that?*

Problem 2.5. *Finally, all the three students are given the yellow-labeled hats. Can they figure out the color of their labels? How?*

Challenge Problem 2.6. *The final exam for the students goes as follows: The entire class of 100 students are placed in a room, and the lights are turned off. A hat is placed on each of their heads, and they are told that at least one hat has a green dot. For the next two hours, the lights will turn on for 30 seconds, and then off for 30 seconds. If at any point a student believes their hat has a green dot, they are supposed to leave the room the next time the lights are off. If a student with a yellow dot leaves the room before the two hours are up, they fail the class.*

If a student with a green dot is still in the room after the two hours are up, they fail the class. Otherwise, the student passes.

Suppose every student has a green dot on their hat. If all the students have successfully learned logic, what will happen over the next two hours?

3 Prison Games

In a strange prison, prisoners are often given the opportunity to play a game to go free early. The games are very different, but share one important rule: prisoners may not communicate once the game starts. However, they are given an opportunity to strategize once they hear the rules.

Problem 3.1. *Two prisoners are each given a coin, and are sent to separate rooms. They must each flip their coin, and attempt to guess the other prisoner's coin. If at least one of them guesses right, they both win, and are set free. If they both guess wrong, they both lose, and must remain in prison for life. How can the prisoners guarantee a win?*

Problem 3.2. *Three prisoners are each given a hat. The hats come in 3 colors (black, white and red) and it is possible for multiple prisoners to have the same colored hat. Each prisoner can only see the hats of the other two prisoners, and has to guess the color of their own. If at least one of them guesses right, all of them win and are set free. If all three guess wrong, they all lose and must remain in prison for life. Find a strategy for the prisoners to guarantee their freedom. Can they still find a strategy that always works if two of them need to guess right for all of them to win?*

Problem 3.3. *A single prisoner is blindfolded, taken into a dark room, and brought to a table with 100 coins on it. They are told that 14 coins are heads, and the rest are tails. The prisoner must then split the coins into two groups such that each group has an equal number of heads in it. They can flip coins over, but they can't at any point look at the coins. How can the prisoner accomplish this task? (This isn't a trick question.)*

Problem 3.4. *20 prisoners are kept in isolated cells. Every day, the warden chooses a completely random prisoner and takes them to a room with two switches, where the prisoner has to flip exactly one switch and leave the room. The switches do not connect to anything, and both start in the OFF position. The warden does not guarantee anything about the choice of prisoner except that eventually, all of the prisoners will have visited the room as many times as they can count. At any time, any prisoner can declare that all of the prisoners have visited the switch room. If they are right, the prisoners win and all of them are set free. If they are wrong, the prisoners lose and remain imprisoned for life. How can the prisoners guarantee a win? (Bonus problem: what if the prisoners don't know the initial state of the switches?)*

Challenge Problem 3.5. *This game uses two prisoners. Before the game begins, the warden places a coin on each square of a chessboard. Each coin is randomly flipped either heads or tails. The warden hides the key to the prison under one coin (it's a very small key). The first prisoner is brought to the chessboard, and is shown where the key is. They may then flip one, and only one, coin from heads to tails, or from tails to heads. The first prisoner is then sent away, and the warden rotates and shifts all the coins slightly (so no information can be communicated via the positions/rotations of the coins). The second prisoner is brought to the chessboard, and can lift one, and only one, coin. If the key is under that coin, the prisoners win, and can escape. Otherwise, they lose. How can the prisoners guarantee a win?*

Challenge Problem 3.6. *100 prisoners will play this game. 100 boxes numbered from 1 to 100 are placed in a room, and 100 slips of paper numbered from 1 to 100 are randomly placed in the box, with 1 paper per box. Each prisoner is also assigned a unique number from 1 to 100. One by one, each prisoner is brought into the room, and may open up to 50 boxes. If the prisoner finds the paper with their own number, they have succeeded. Otherwise, they have failed. The prisoner then leaves the room, all the boxes are closed again, and a new prisoner is brought into the room. If every prisoner succeeds, they all win. However, if even one prisoner fails, they all lose. What is a good strategy the prisoners can employ, if they are not allowed to communicate once the game starts. (Note: Unlike the other games, the prisoners can't guarantee a win. However, they can significantly improve their chances from a random guess).*