Graph Theory and Combinatorial Optimization: Solutions

Sunny Liang

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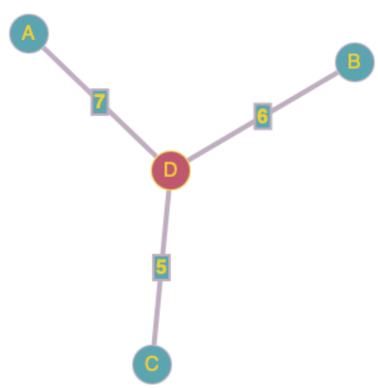
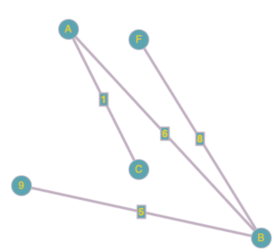
LaTeX source: `https://www.overleaf.com/read/wpkmjxdkgbfd`

1. Sum of degrees is twice the number of edges.
2. Use answer to previous problem. Clearly there must be an even number odd-degree vertices or else the sum of degrees would be odd.

   Students should understand that each vertex represents a person and each handshake is an edge. Thus, each person shakes hands a number of times equal to the degree of their vertex.
3. No.
4. There is an Eulerian cycle on a connected graph if and only if there are 0 or 2 nodes.
5. $\frac{(n-1)!}{2}$
6. Weight = 18



7. Weight = 20



8. MSTs are unique to a graph iff all the weights are unique.

9. Draw an edge between any unconnected vertices with sufficiently large weight. You can easily guarantee that the weight is large enough by making it larger than the sum of all other weights.

10. Prim's algorithm: keep taking the lowest weight edge of a graph until all nodes are connected. If there are multiple edges of the same weight, take whichever one. Time complexity is at most $O(n^2)$ where n is the number of vertices.

11. ADCBA or ACDBA both have length 30.

12. Order does not matter since a complete tour of all the vertices by definition visits all vertices symmetrically.

13. Figure 3b has edge weights that violate the triangle inequality. That is, given a triangle $\triangle ABC$, $\overline{AB} + \overline{BC} \geq \overline{AC}$. This always holds in Euclidean geometry but not necessarily other geometries. What if your cities are not on the surface of a torus? What if there are mountains in between your cities?

14. $\frac{(n-1)!}{2}$

15. Plugging into formula from previous problem and using WolframAlpha:

    21 nodes = 38.57 years, 29 nodes = 4.834 trillion years.

16. This is open-ended, most likely answers are explored in the next section.

17. You have to compare $(n-i)$ edges for $i = 1$ to n so $\sum_{i=1}^{n}(n-i) = \frac{n(n-1)}{2}$ comparisons. Since we don't care about coefficients and lower-order terms, we just say that this is $O(n^2)$. The most important part of this question is the student's reasoning.

18. There are infinitely many answers to this, here's a simple example from S.K. Basu's *Design Methods and Analysis of Algorithms*:

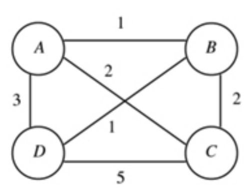
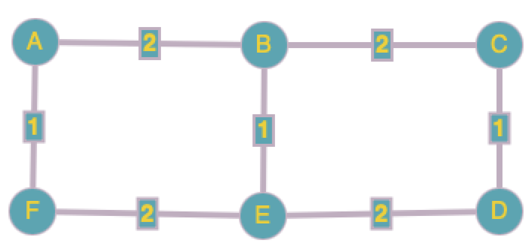
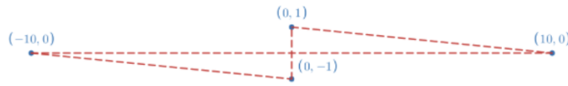


FIGURE 4.7 Example graph for TSP.
From A, the greedy cycle is $ABDCA$ of length 9, while $ACBDA$ has length 8.

Another simple example:

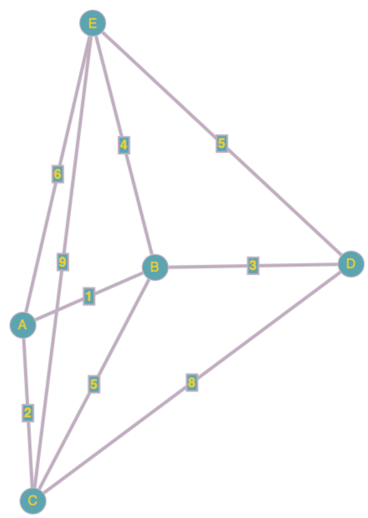


19. Again, there are infinitely many answers to this, here's a simple example from a Mathematics Stack Exchange post (`https://math.stackexchange.com/q/4404052`):



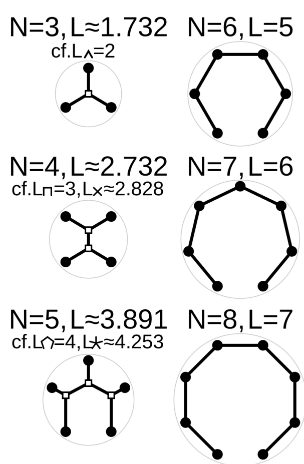
20. By similar reasoning as in Problem 17, we have $O(n^2)$, though, using clever sorting algorithms we can improve the average case to be closer to $O(n \log n)$.

21. By carefully selecting weights, you can construct something like this which is designed to reward Multi-fragment's consideration of non-adjacent edges.



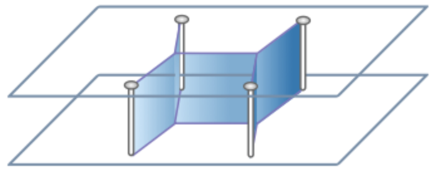
22. From Wikipedia: By Cmglee - Own work, CC BY-SA 4.0,

    https://commons.wikimedia.org/w/index.php?curid=83330247.



23. See figure above. You should notice that Steiner trees never make angles less than 120°. You can try to think of why by analyzing the Steiner tree on a triangle as you vary its height.

24. Here's one way you could do it from David Wakeham (`https://arxiv.org/pdf/2008.09611.pdf`):



25. Taken from Nakul and Andrea's handout on geometric optima for Advanced 2 2022-2023, also see Exercise 2.9 in David Wakeham's notes above.

    (a) Consider the area of the triangle. By connecting AP, BP, and CP, we see that the area should be $\frac{1}{2}(AB \cdot DP + BC \cdot EP + AC \cdot FP) = \frac{AB}{2}(DP + PF + PE)$. Hence, $PD + PE + PF$is constant for every P.

    (b) For equilateral triangle, the circumcentre is the obvious answer. When the triangle is not equilateral, the answer is the Toricelli point, which is the point where $\angle APB = \angle BPC = \angle CPA = 2\pi/3$. One can construct this point by drawing an equilateral triangle on each of two arbitrarily chosen sides, say AB and BC, and then connecting the vertex of each equilateral not on $\triangle ABC$ to the opposite side, for example, connect the vertex of the equilateral triangle with side AB to C, and the intersection of the two lines would be the Toricelli point.

26. `https://youtu.be/GiDsjIBOVoA?si=Jhw7ACw0rBzgX-JI&t=726`

27. ditto