
Lambda Calculus

Prepared by Mark on April 3, 2024

Beware of the Turing tar pit, in which everything is possible but nothing of interest is easy.

Alan Perlis, *Epigrams of Programming*, #54

Part 1: Introduction

Lambda calculus is a model of computation, much like the Turing machine. As we're about to see, it works in a fundamentally different way, which has a few practical applications we'll discuss at the end of class.

A lambda function starts with a lambda (λ), followed by the names of any inputs used in the expression, followed by the function's output.

For example, $\lambda x.x + 3$ is the function $f(x) = x + 3$ written in lambda notation.

Let's dissect $\lambda x.x + 3$ piece by piece:

- “ λ ” tells us that this is the beginning of an expression.
 λ here doesn't have a special value or definition;
it's just a symbol that tells us “this is the start of a function.”
- “ λx ” says that the variable x is “bound” to the function (i.e, it is used for input).
Whenever we see x in the function's output, we'll replace it with the input of the same name.
This is a lot like normal function notation: In $f(x) = x + 3$, (x) is “bound” to f , and we replace every x we see with our input when evaluating.
- The dot tells us that what follows is the output of this expression.
This is much like $=$ in our usual function notation:
The symbols after $=$ in $f(x) = x + 3$ tell us how to compute the output of this function.

Problem 1:

Rewrite the following functions using this notation:

- $f(x) = 7x + 4$
- $f(x) = x^2 + 2x + 1$

To evaluate $\lambda x.x + 3$, we need to input a value:

$$(\lambda x.x + 3) 5$$

This is very similar to the usual way we call functions: we usually write $f(5)$.

Above, we define our function f “in-line” using lambda notation, and we omit the parentheses around 5 for the sake of simpler notation.

We evaluate this by removing the “ λ ” prefix and substituting 3 for x whenever it appears:

$$(\lambda x.x + 3) 5 = 5 + 3 = 8$$

Problem 2:

Evaluate the following:

- $(\lambda x.2x + 1) 4$
- $(\lambda x.x^2 + 2x + 1) 3$
- $(\lambda x.(\lambda y.9y)x + 3) 2$

Hint: This function has a function inside, but the evaluation process doesn’t change. Replace all x with 2 and evaluate again.

As we saw above, we denote function application by simply putting functions next to their inputs. If we want to apply f to 5, we write “ $f 5$ ”, without any parentheses around the function’s argument.

You may have noticed that we’ve been using arithmetic in the last few problems. This isn’t fully correct: addition is not defined in lambda calculus. In fact, nothing is defined: not even numbers! In lambda calculus, we have only one kind of object: the function. The only action we have is function application, which works by just like the examples above.

Don’t worry if this sounds confusing, we’ll see a few examples soon.

Definition 3:

The first “pure” functions we’ll define are I and M :

- $I = \lambda x.x$
- $M = \lambda x.xx$

Both I and M take one function (x) as an input.

I does nothing, it just returns x .

M is a bit more interesting: it applies the function x on a copy of itself.

Also, note that I and M don’t have a meaning on their own. They are not formal functions.

Rather, they are abbreviations that say “write $\lambda x.x$ whenever you see I .”

Problem 4:

Reduce the following expressions.

Hint: Of course, your final result will be a function.

Functions are the only objects we have!

- $I I$
- $M I$
- $(I I) I$
- $\left(\lambda a.(a (a a)) \right) I$
- $\left(\lambda a.(\lambda b.a) M \right) I$

Example Solution**Solution for $(I I)$:**

Recall that $I = \lambda x.x$. First, we rewrite the left I to get $(\lambda x.x) I$.

Applying this function by replacing x with I , we get I :

$$I I = (\lambda x.x) I = I$$


In lambda calculus, functions are left-associative:

$(f g h)$ means $((f g) h)$, not $(f (g h))$

As usual, we use parentheses to group terms if we want to override this order: $(f (g h)) \neq ((f g) h)$

In this handout, all types of parentheses ($()$, $[\]$, etc) are equivalent.

Problem 5:

Rewrite the following expressions with as few parentheses as possible, without changing their meaning or structure. Remember that lambda calculus is left-associative.

- $(\lambda x.(\lambda y.\lambda z.((xz)(yz))))$
- $((ab)(cd))((ef)(gh))$
- $(\lambda x.((\lambda y.(yx))(\lambda v.v)z)u)(\lambda w.w)$

Definition 6: Equivalence

We say two functions are *equivalent* if they differ only by the names of their variables:

$$I = \lambda a.a = \lambda b.b = \lambda \heartsuit.\heartsuit = \dots$$

Definition 7:

Let $K = \lambda a.(\lambda b.a)$. We'll call K the "constant function function."

Problem 8:

That's not a typo. Why does this name make sense?

Hint: What is $K x$?

Problem 9:

Show that associativity matters by evaluating $((M K) I)$ and $(M (K I))$.

What would $M K I$ reduce to?

Currying:

In lambda calculus, functions are only allowed to take one argument.

If we want multivariable functions, we'll have to emulate them through *currying*¹

The idea behind currying is fairly simple: we make functions that return functions.

We've already seen this on the previous page: K takes an input x and uses it to construct a constant function. You can think of K as a "factory" that constructs functions using the input we provide.

Problem 10:

Let $C = \lambda f. [\lambda g. (\lambda x. [f(g(x))])]$. For now, we'll call it the "composer."

Note: We could also call C the "right-associator." Why?

C has three "layers" of curry: it makes a function (λg) that makes another function (λx) .

If we look closely, we'll find that C pretends to take three arguments.

What does C do? Evaluate $(C a b x)$ for arbitrary expressions a, b , and x .

Hint: Evaluate $(C a)$ first. Remember, function application is left-associative.

Problem 11:

Using the definition of C above, evaluate $C M I \star$

Then, evaluate $C I M I$

Note: \star represents an arbitrary expression. Treat it like an unknown variable.

As we saw above, currying allows us to create multivariable functions by nesting single-variable functions. You may have notice that curried expressions can get very long. We'll use a bit of shorthand to make them more palatable: If we have an expression with repeated function definitions, we'll combine their arguments under one λ .

For example, $A = \lambda f. [\lambda a. f(f(a))]$ will become $A = \lambda f a. f(f(a))$

Problem 12:

Rewrite $C = \lambda f. \lambda g. \lambda x. (g(f(x)))$ from Problem 10 using this shorthand.

Remember that this is only notation. **Curried functions are not multivariable functions, they are simply shorthand!** Any function presented with this notation must still be evaluated one variable at a time, just like an un-curried function. Substituting all curried variables at once will cause errors.

¹After Haskell Brooks Curry? a logician that contributed to the theory of functional computation.

²There are three programming languages named after him: Haskell, Brook, and Curry. Two of these are functional, and one is an oddball GPU language last released in 2007.

Problem 13:

Let $Q = \lambda abc.b$. Reduce $(Q a c b)$.

Hint: You may want to rename a few variables.

The a, b, c in Q are different than the a, b, c in the expression!

Problem 14:

Reduce $((\lambda a.a) \lambda bc.b) d \lambda eg.g$

Part 2: Combinators

Definition 15:

A *free variable* in a λ -expression is a variable that isn't bound to any input. For example, b is a free variable in $(\lambda a.a) b$.

Definition 16: Combinators

A *combinator* is a lambda expression with no free variables.

Notable combinators are often named after birds.³ We've already met a few:

The *Idiot*, $I = \lambda a.a$

The *Mockingbird*, $M = \lambda f.f f$

The *Cardinal*, $C = \lambda f g x.(f(g(x)))$ The *Kestrel*, $K = \lambda a b.a$

Problem 17:

If we give the Kestrel two arguments, it does something interesting:

It selects the first and rejects the second.

Convince yourself of this fact by evaluating $(K \heartsuit \star)$.

Problem 18:

Modify the Kestrel so that it selects its **second** argument and rejects the first.

Problem 19:

We'll call the combinator from Problem 18 the *Kite*, KI .

Show that we can also obtain the kite by evaluating $(K I)$.

Part 3: Boolean Algebra

The Kestrel selects its first argument, and the Kite selects its second. Maybe we can somehow put this “choosing” behavior to work...

Let $T = K = \lambda ab.a$

Let $F = KI = \lambda ab.b$

Problem 20:

Write a function NOT so that $(\text{NOT } T) = F$ and $(\text{NOT } F) = T$.

Hint: What is $(T \heartsuit \star)$? How about $(F \heartsuit \star)$?

Problem 21:

How would “if” statements work in this model of boolean logic?

Say we have a boolean B and two expressions E_T and E_F . Can we write a function that evaluates to E_T if B is true, and to E_F otherwise?

Problem 22:

Write functions AND, OR, and XOR that satisfy the following table.

A	B	$(\text{AND } A B)$	$(\text{OR } A B)$	$(\text{XOR } A B)$
F	F	F	F	F
F	T	F	T	T
T	F	F	T	T
T	T	T	T	F

Problem 23:

To complete our boolean algebra, construct the boolean equality check EQ. What inputs should it take? What outputs should it produce?

Part 4: Numbers

Since the only objects we have in λ calculus are functions, it's natural to think of quantities as *adverbs* (once, twice, thrice,...) rather than *nouns* (one, two, three ...)

We'll start with zero. If our numbers are *once*, *twice*, and *twice*, it may make sense to make zero *don't*. Here's our *don't* function: given a function and an input, don't apply the function to the input.

$$0 = \lambda f a.a$$

If you look closely, you'll find that 0 is equivalent to the false function F .

Problem 24:

Write 1, 2, and 3. We will call these *Church numerals*.⁴

Note: This problem read aloud is "Define *once*, *twice*, and *thrice*."

Problem 25:

What is $(4 I) \star$?

Problem 26:

What is $(3 NOT T)$?

How about $(8 NOT F)$?

⁴after Alonzo Church, the inventor of lambda calculus and these numerals. He was Alan Turing's thesis advisor.

Problem 27:

Peano's axioms state that we only need a zero element and a "successor" operation to build the natural numbers. We've already defined zero. Now, create a successor operation so that $1 := S(0)$, $2 := S(1)$, and so on.

Hint: A good signature for this function is $\lambda n f a$, or more clearly $\lambda n. \lambda f a$. Do you see why?

Problem 28:

Verify that $S(0) = 1$ and $S(1) = 2$.

Assume that only Church numerals will be passed to the functions in the following problems. We make no promises about their output if they're given anything else.

Problem 29:

Define a function ADD that adds two Church numerals.

Problem 30:

Design a function MULT that multiplies two numbers.

Hint: The easy solution uses ADD, the elegant one doesn't. Find both!

Problem 31:

Define the functions Z and NZ . Z should reduce to T if its input was zero, and F if it wasn't. NZ does the opposite. Z and NZ should look fairly similar.

Problem 32:

Design an expression PAIR that constructs two-value tuples.

For example, say $A = \text{PAIR } 1 \ 2$. Then,

$(A \ T)$ should reduce to 1 and $(A \ F)$ should reduce to 2.

From now on, I'll write $(\text{PAIR } A \ B)$ as $\langle A, B \rangle$.

Like currying, this is only notation. The underlying logic remains the same.

Problem 33:

Write a function H , which we'll call "shift and add."

It does exactly what it says on the tin:

Given an input pair, it should shift its second argument left, then add one.

$H \langle 0, 1 \rangle$ should reduce to $\langle 1, 2 \rangle$

$H \langle 1, 2 \rangle$ should reduce to $\langle 2, 3 \rangle$

$H \langle 10, 4 \rangle$ should reduce to $\langle 4, 5 \rangle$

Problem 34:

Design a function D that un-does S . That means

$D(1) = 0$, $D(2) = 1$, etc. $D(0)$ should be zero.

Hint: H will help you make an elegant solution.

Part 5: Recursion

Say we want a function that computes the factorial of a positive integer. Here's one way we could define it:

$$x! = \begin{cases} x \times (x - 1)! & x \neq 0 \\ 1 & x = 0 \end{cases}$$

We cannot re-create this in lambda calculus, since we aren't given a way to recursively call functions.

One could think that $A = \lambda a.A$ a is a recursive function. In fact, it is not.

Remember that such "definitions" aren't formal structures in lambda calculus.

They're just shorthand that simplifies notation.

Problem 35:

Write an expression that resolves to itself.

Hint: Your answer should be quite short.

This expression is often called Ω , after the last letter of the Greek alphabet.

Ω useless on its own, but it gives us a starting point for recursion.

Definition 36:

This is the *Y-combinator*. You may notice that it's just Ω put to work.

$$Y = \lambda f.(\lambda x.f(x x))(\lambda x.f(x x))$$

Problem 37:

What does this thing do?

Evaluate Yf .

Part 6: Challenges

Do Problem 38 first, then finish the rest in any order.

Problem 38:

Design a recursive factorial function using Y .

Problem 39:

Design a non-recursive factorial function.

This one is easier than Problem 38, but I don't think it will help you solve it.

Problem 40:

Solve Problem 34 without using H .

In Problem 34, we created the “decrement” function.

Problem 41:

Using pairs, make a “list” data structure. Define a GET function, so that GET $L n$ reduces to the n th item in the list. GET $L 0$ should give the first item in the list, and GET $L 1$, the *second*.

Lists have a defined length, so you should be able to tell when you're on the last element.

Problem 42:

Write a lambda expression that represents the Fibonacci function:

$$f(0) = 1, f(1) = 1, f(n + 2) = f(n + 1) + f(n).$$

Problem 43:

Write a lambda expression that evaluates to T if a number n is prime, and to F otherwise.

Problem 44:

Write a function MOD so that $(\text{MOD } a \ b)$ reduces to the remainder of $a \div b$.

Problem 45: Bonus

Play with *Lamb*, an automatic lambda expression evaluator.

<https://git.betalupi.com/Mark/lamb>