Introduction to Inequalities I

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Ice-breaker.

In this packet, we're going to learn how to work with a few common inequalities. A lot of higher-level math involves approximating quantities of interest instead of calculating them explicitly. For instance, consider the $\sqrt{x}$ function on your phone's calculator. Your phone doesn't naturally know how to compute a square root exactly, so it needs to come up with a sequence of real numbers that converge to $\sqrt{x}$. Inequalities are used here to quantify how good the iteration is – for instance, an inequality like this might be used:

$$\text{(error of the } n\text{th iterate)} \leq \ldots$$

Sometimes, computers will use a randomized algorithm instead of a deterministic one. For instance, one way to estimate π is to inscribe a unit circle inside a square with side length 2 and uniformly randomly pick points from the square.

Exercise 1. *Warmup: Given the proportion p of randomly chosen points which land inside the inscribed circle, how would you estimate π?*

In any case, the error of this algorithm can be bounded like (given that we sample n points:

$$\text{(probability that the error is at least ...)} \leq \text{...}$$

Exercise 2. *If $a \leq b$ and $c \geq 0$, then is $ca \leq cb$? Why or why not?*

Exercise 3. *If $a \leq b$ and $c \leq 0$, then is $ca \leq cb$? Why or why not?*

For this worksheet, consider the statement that for all nonnegative $x_1, \cdots, x_n$, one has:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

This is called the arithmetic mean-geometric mean (AM-GM) inequality. Both quantities represent averages of different kinds.

Exercise 4. *Try computing the AM and GM of the sets $\{1, 3, 5\}$ and $\{2, 3, 5, 7, 9\}$ respectively. Try a few more examples of your own creation. What is qualitatively different about the AM and the GM? When would we want to use one over the other to represent the average of a set of numbers?*

Exercise 5. *Is the AM-GM inequality true when $n = 1$? Why?*

Exercise 6. *Is the AM-GM inequality true when $n = 2$? Hint: consider the expression $(\sqrt{x_1} - \sqrt{x_2})^2$ (which is obviously nonnegative). Does this give you an inequality you can use?*

The $n = 2$ case of the AM-GM inequality is very important, so please make sure you convince yourself of this before moving on.

Exercise 7. *The Cauchy-Schwarz inequality (when $n = 2$) is $x_1y_1 + x_2y_2 \leq \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)}$ for real numbers x_1, x_2 and y_1, y_2. Assume the Cauchy-Schwarz inequality is true, and derive the $n = 2$ case of AM-GM as a consequence by choosing x_1, x_2, y_1, y_2 appropriately.*

Exercise 8. *Use the AM-GM inequality to prove that* $x + \frac{1}{x} \geq 2$ *for positive* x*. What does* x *need to be in order to achieve equality in the above inequality?*

Exercise 9. *For those of you familiar with logarithms, a topic we covered last year. Suppose some nonnegative* $x_1, \cdots, x_n$*. Using properties of the natural logarithm,* $\ln$*, show that the AM of the natural logarithms of the* x_i*, i.e* $\frac{\ln x_1 + \cdots + \ln x_n}{n}$ *is the natural logarithm of the GM.*

Exercise 10. *Let* x *be some fixed real number and* N *be some fixed positive integer. What is the AM of* $\log x^1, \log x^2, \cdots, \log x^N$ *in terms of* x *and* N*. Hint: You may want to use the formula for the sum of the first* N *positive integers.*

Exercise 11. *If $x + y = 6$, what is the biggest possible value for xy?*

Exercise 12. *If $xy = 16$, what is the smallest possible value for $x + y$?*

Exercise 13. *Prove that for every nonnegative x, y and z, one has $x^2 + y^2 + z^2 \geq xy + xz + yz$.*

Exercise 14. *Prove that for every nonnegative x, y and z, one has $x^3 + y^3 + z^3 \geq 3xyz$.*

Exercise 15. *Can you use the results of the previous two exercises to prove the $n = 3$ case of AM-GM? Hint: Take the difference of both sides of the inequality and factor out $(x + y + z)$.*

It turns out that proving the general AM-GM inequality for any positive integer n and is not very easy, and so Cauchy actually proved it for each n by using the following trick which is now known as Cauchy induction.

Exercise 16. *Assume that the $n = 2$ case of AM-GM is true (since we have already proven it) and also assume that the $n = k$th case of AM-GM is true. Use these two assumptions to show that the $n = 2k$th case of AM-GM is true. Hint: consider $\frac{a_1+\cdots+a_k}{k}$ and $\frac{a_{k+1}+\cdots+a_{2k}}{k}$. What can you tell me about each of them individually using the $n = k$ case of AM-GM? Also, what can you tell me about the process of combining them into one AM expression?*

Exercise 17. *Prove that the nth case AM-GM inequality implies the $n - 1$st case AM-GM inequality, i.e* *in notation* $\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n} \implies \frac{x_1+x_2+\cdots+x_{n-1}}{n-1} \geq \sqrt[n-1]{x_1x_2\cdots x_{n-1}}$.

Hint: Let $a_n = \frac{a_1+\cdots+a_{n-1}}{n-1}$

Exercise 18. *Prove that the AM-GM inequality is true for every* n*.*

We say that a function f from $\mathbb{R}$ to $\mathbb{R}$ is midpoint-convex if $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$ for all x, y in $\mathbb{R}$.

Exercise 19. *Use Cauchy induction to prove that, if f is midpoint convex, then $f(\frac{x_1+\cdots+x_n}{n}) \leq \frac{f(x_1)+\cdots+f(x_n)}{n}$ for all $n \geq 1$ and $x_1, \cdots, x_n$ in $\mathbb{R}$.*

a) Do the $n \to 2n$ step;

b) Do the $n \to n-1$ step.

Exercise 20. *a) Prove that the function $f(x) = x^2$ is midpoint-concave.*

b) For positive $x_1, \cdots, x_n$ use the previous problem to prove the AM-QM inequality:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}$$

Remark: QM stands for Quadratic Mean.