Olympiads Week 4: Polynomials

ORMC

10/8/23

Remember to *write down your solutions, as proofs.* You don't have to start by writing out a full proof to every problem you try, but once you've solved a problem or two, take a few minutes to write out a proof as if this was being graded at an Olympiad.

1 Book Problems from Putnam and Beyond

In class, I explained how to use some guess-and-check or a determinant argument to come up the following factorization:

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc).$$

This factorization helps us with the first four problems.

Problem 1.1. Show that if x, y, z are distinct real numbers,

$$\sqrt[3]{x-y} + \sqrt[3]{y-z} + \sqrt[3]{z-x} \neq 0.$$

Problem 1.2. What are the real solutions to

$$\sqrt[3]{x-1} + \sqrt[3]{x} + \sqrt[3]{x+1} = 0?$$

Problem 1.3. Find all triples of positive integers x, y, z such that

$$x^3 + y^3 + z^3 - 3xyz = p,$$

where p is a prime greater than 3.

Problem 1.4. Let a, b, c be distinct positive integers such that $ab + bc + ca \geq 3k^2 - 1$, where k is also a positive integer. Show that

$$a^3 + b^3 + c^3 \geq 3(abc + 3k).$$

Problem 1.5. If $n \geq 0$ is an integer, show that the following can't both be perfect cubes:

$$n + 3, n^2 + 3n + 3$$

Problem 1.6. Show that

$$\sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}} = 4.$$

Problem 1.7. Let $P(x, y, z)$ be a polynomial. Show that

$$P(x, y, z) + P(y, z, x) + P(z, x, y) - P(x, z, y) - P(y, x, z) - P(z, y, x)$$

is divisible by $(x - y)(y - z)(x - z)$.

2 Competition Problems

Problem 2.1 (HMMT 2007)**.** The complex numbers $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

Problem 2.2 (BAMO 2012 Problem 7)**.** Find all nonzero polynomials $P(x)$ with integer coefficients that satisfy the following property: whenever a and b are relatively prime integers, then $P(a)$ and $P(b)$ are relatively prime as well. Prove that your answer is correct. (Two integers are relatively prime if they have no common prime factors. For example, -70 and 99 are relatively prime, while -70 and 15 are not relatively prime.)

Problem 2.3 (BAMO 2017 Problem 5)**.** Call a number T persistent if the following holds: Whenever a, b, c, d are real numbers different from 0 and 1 such that

$$a + b + c + d = T$$

and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = T,$$

we also have

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} + \frac{1}{1-d} = T.$$

What numbers are persistent?

Problem 2.4 (USAMO 1995 Problem 4)**.** Suppose $q_0, q_1, q_2, \ldots$ is an infinite sequence of integers satisfying the following two conditions:

- $m - n$ divides $q_m - q_n$ for $m > n \geq 0$,
- there is a polynomial P such that $|q_n| < P(n)$ for all n.

Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.

Problem 2.5 (Putnam 2019 Problem B5)**.** Let F_m be the mth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \ldots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.

2.1 Eisenstein's Criterion

If $f(x)$ is an integer polynomial, we say that it's *irreducible* when there are no nonconstant polynomials $g(x)$ and $h(x)$ with $f(x) = g(x)h(x)$. As a hint for an IMO problem, let's prove a tool called *Eisenstein's Criterion* that helps determine when integer polynomials are irreducible.

Problem 2.6. Prove *Eisenstein's Criterion*:

Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be an integer polynomial, and let p be a prime such that

- For $i < n$, $p | a_i$
- $p \nmid a_n$
- $p^2 \nmid a_0$.

Then $f(x)$ is irreducible.

Problem 2.7 (IMO 1993 Problem 1)**.** Let $n > 1$ be an integer. Prove that there are no nonconstant polynomials $g(x)$ and $h(x)$ with integer coefficients such that

$$g(x)h(x) = x^n + 5x^{n-1} + 3.$$