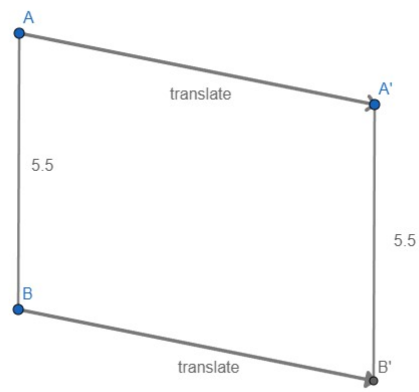
GROUP THEORY VIA SYMMETRY

MAX STEINBERG FOR THE OLGA RADKO MATH CIRCLE
ADVANCED 2

1. Warmup: Symmetries

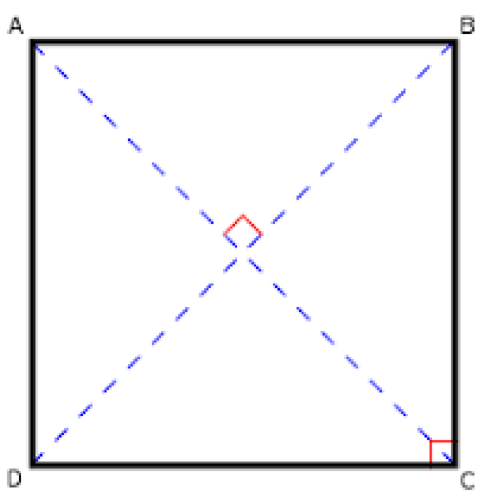
We say a plane transformation is a **isometry** if it doesn't change the distance between two points. For example, a translation is an isometry:



Problem 1. Are the following plane transformations isometries? Why or why not?

(1) A rotation by some angle.
(2) The map of the plane which takes the point (x, y) to $(2x, 2y)$.
(3) Reflections about some line.
(4) The map of the plane which takes the point (x, y) to $(x + y, y)$.

We define a **symmetry** of a set of points in the plane to be a plane isometry such that every point in the set maps to another point in the set (possibly the same point). So for example, if we have a square, a symmetry of the square might be a reflection across one of the diagonals. The identity transformation counts as a symmetry.



Problem 2. Find all of the symmetries of a square. Make sure to consider all possible translations, rotations, and reflections. *(Hint: You can draw on the picture above!)*

Problem 3. Find all of the symmetries of a snowflake.

In three dimensions, all of our definitions still work as you would expect. We still only need to consider translations, rotations, and reflections.

Problem 4. Find all the symmetries of a cube.

2. Groups

A **group**, roughly speaking, is a set of objects together with a **binary operation**: an operation that can be applied to two elements of the set and returns an element of the set. We typically write a group like $(\mathbb{Z}, +)$, where the first item is the set ($\mathbb{Z}$) and the second is the operation (addition, $+$). There are some **group axioms** that every group must follow in order to be considered a group. Let G be a set and $+$ be a binary operation on G that returns an element of G (ie. $+ : G \times G \to G$). Then if all of the following four axioms are true, we says $(G, +)$ is a group:

(1) There must be an identity element. That is, there is some element $e \in G$ so that for every $g \in G$, $e + g = g + e = g$.
(2) G must be closed under its binary operation. That is, for every $a, b \in G$, it must be the case that $a + b \in G$.
(3) The binary operation must be associative. That is, for every $a, b, c \in G$, $(a + b) + c = a + (b + c)$.
(4) Every element must have an inverse. That is, for every $a \in G$, there is some $b \in G$ so that $a + b = b + a = e$.

You may notice $x + y$ may not equal $y + x$ (ie. our binary operation need not be commutative). If $x + y = y + x$ for every x, y in our group, our group is called "abelian."

Problem 5. Which of the following are groups? Why or why not?

(1) $(\mathbb{Z}, +)$
(2) $(\mathbb{Z}, \cdot)$
(3) $(\mathbb{R}, +)$
(4) $(\mathbb{R}, \cdot)$
(5) $(\mathbb{R} \setminus \{0\}, +)$
(6) $(\mathbb{R} \setminus \{0\}, \cdot)$

Problem 6. Groups don't always have to be groups of numbers. Verify that the following set with a binary operation is a group (ie. go through each of the four axioms and check that they are satisfied): let $G = \{\text{odd}, \text{even}\}$ with the operator $+$ defined by $\text{odd} + \text{odd} = \text{even}$, $\text{odd} + \text{even} = \text{odd}$, $\text{even} + \text{odd} = \text{odd}$, $\text{even} + \text{even} = \text{even}$.

Problem 7. Are there any groups with 0 elements? How about 1 element? 2 elements?

Problem 8. Let us take the set $\{e, a, b, a^2, b^2, \dots\}$ as our set, and define our binary operation as follows:

$\times$	e	a	b
e	e	a	b
a	a	a^2	e
b	b	e	b^2

(and continuing on, so that, for example, $a^2 \cdot a = a^3$, etc.) Does our set with this binary operation form a group? Why or why not?

Problem 9. Let $(G, \cdot)$ be a group. Is the identity of G unique? Let $a \in G$. Is the inverse of a unique? Prove your answer.

Problem 10. Prove $(a^{-1})^{-1} = a$ for any $a \in G$.

3. Group Presentations

As of right now, it might seem difficult to write down exactly what a group is. It's not easy to say or write "the set of $\{e, a, b, a^2, b^2, \dots\}$ with this multiplication table." So let's come up with a method of writing down certain kinds of groups in ways that are easy to deal with.

Instead of writing out the list of elements in our group along with a multiplication table (as this is extremely repetitive and not useful), we will work with **group presentations**. Group presentations have two parts: the alphabet and the rules. Let's look at an example first with just an alphabet:

$$\langle \underbrace{a, b}_{\text{alphabet}} \rangle$$

this group is composed of all words built in the alphabet $\{a, b\}$, with the group operation being concatenation (which we will denote by $\oplus$). So for example, this group has the element aba, and $aba \oplus bab = ababab$. In this group, we will use e to denote the identity (so $e \oplus ab = ab$ not eab). Further, elements are required to have inverses, so we will allow a^{-1} and b^{-1} to be used in the alphabet as well, with the requirement that any time a sits next to a^{-1} they are both removed. So

$$a^{-1}bb^{-1}a \Rightarrow a^{-1} \underbrace{bb^{-1}}_{\text{cancelling } b \text{ and its inverse}} a \Rightarrow \underbrace{a^{-1}a}_{\text{cancelling } a \text{ and its inverse}} \Rightarrow e$$

Now let's add some rules.

$$\langle \underbrace{a, b}_{\text{alphabet}} \mid \underbrace{a^2, b^3}_{\text{rules}} \rangle$$

Here, we say that any time a^2 appears, we remove it. Similarly, any time b^3 appears, we remove it.

Problem 11. Simplify $ab^2b^{-1}b^{-1}ab^2aa^{-1}b$.

Problem 12. Verify that the group $\langle a, b | a^2, b^3 \rangle$ is in fact a group. (That is, verify the group axioms from page 1.)

The **order** of a group, denoted by $|G|$, is the size of G as a set. This is only a useful idea if $|G|$ is finite, in which case G is known as a **finite group**.

Problem 13. Find the order of $\langle a, b | a^2, b^2, aba^{-1}b^{-1} \rangle$. (Don't forget the identity element!)

Problem 14. (Challenge problem) Explain why any group presentation satisfies the group axioms.

Problem 15. (Challenge problem) Pick two positive integers p and q. What is the order of $\langle a | a^p, a^q \rangle$?

4. Symmetries

Now that we got through all that abstract nonsense, let's do some more geometry.

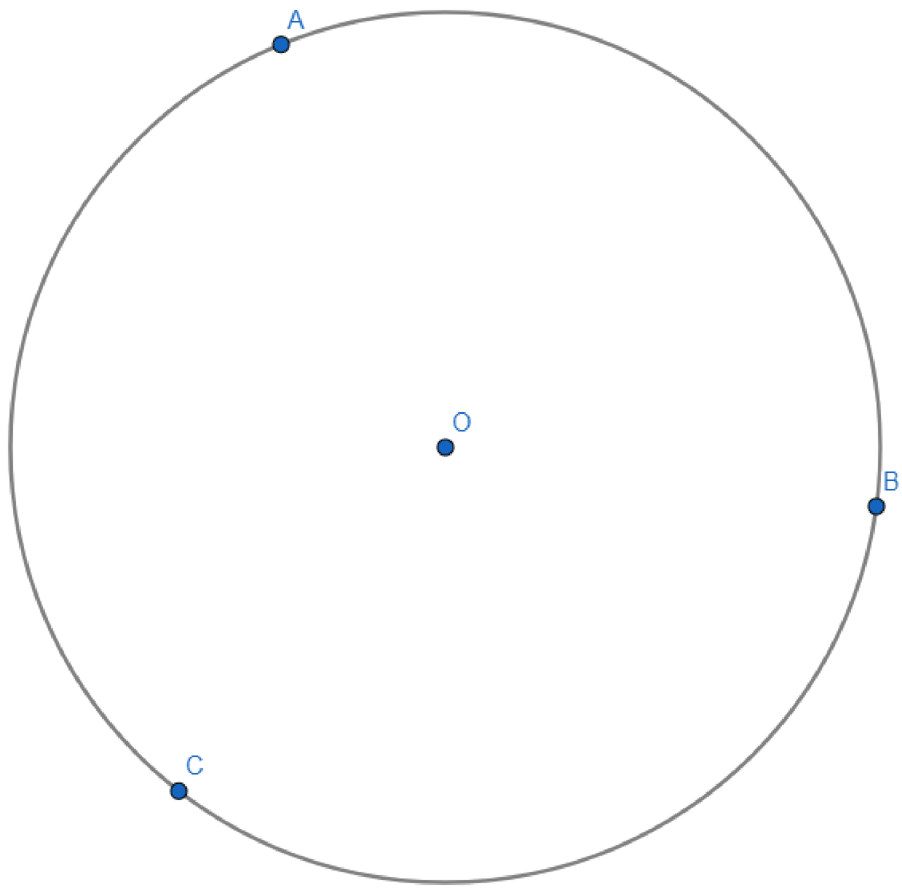


Figure 1

Let's think about how these three points, A, B, C, are symmetric. There is reflectional symmetry, but we will ignore it for now and only focus on the threefold rotational symmetry. How could we *represent* this symmetry? As you may have guessed, we can form a group out of this symmetry. A **symmetry** on a set of points in a plane is a plane isometry which leaves the points fixed. That is a lot of complicated words to say a symmetry is a rotation, reflection, or translation, which moves the points to other points within the set (or to the same point). Thus, under this definition, the identity transformation is a symmetry. We can denote this as e. Furthermore, we can rotate by $\frac{2\pi}{3}$ about O to send $A \to B, B \to C, C \to A$, so this is also a symmetry. We can denote this r (for "rotation").

Problem 16. If we define a binary operation as composition (eg. $e \times e = e \circ e$), write out our multiplication table for $\{e, r, r^2\}$.

$\times$	e	r	r^2
e			
r			
r^2			

Problem 17. What is r^3? With this knowledge, how can we write a presentation for the rotational symmetry group of this figure above?

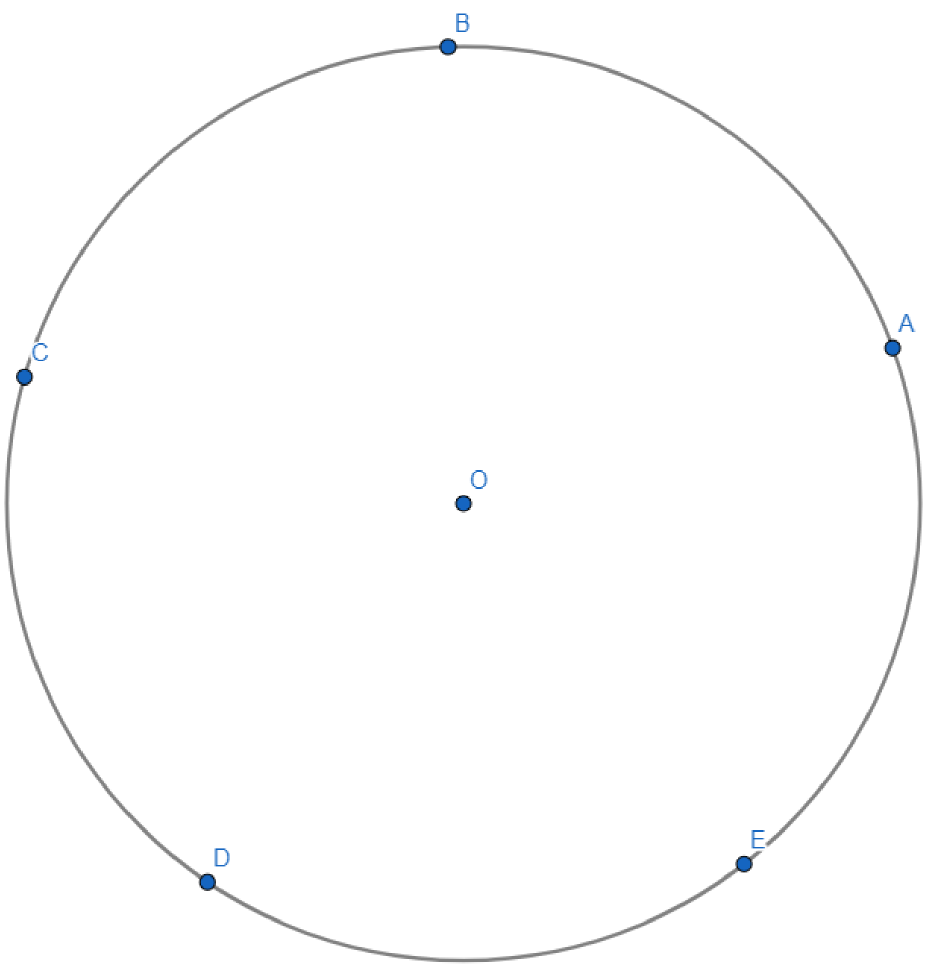


Figure 2

Problem 18. Write the rotational symmetry group for Figure 2.

Problem 19. (Challenge problem) Write the full symmetry group for Figure 1 on a circle (make sure to include the reflectional symmetry).

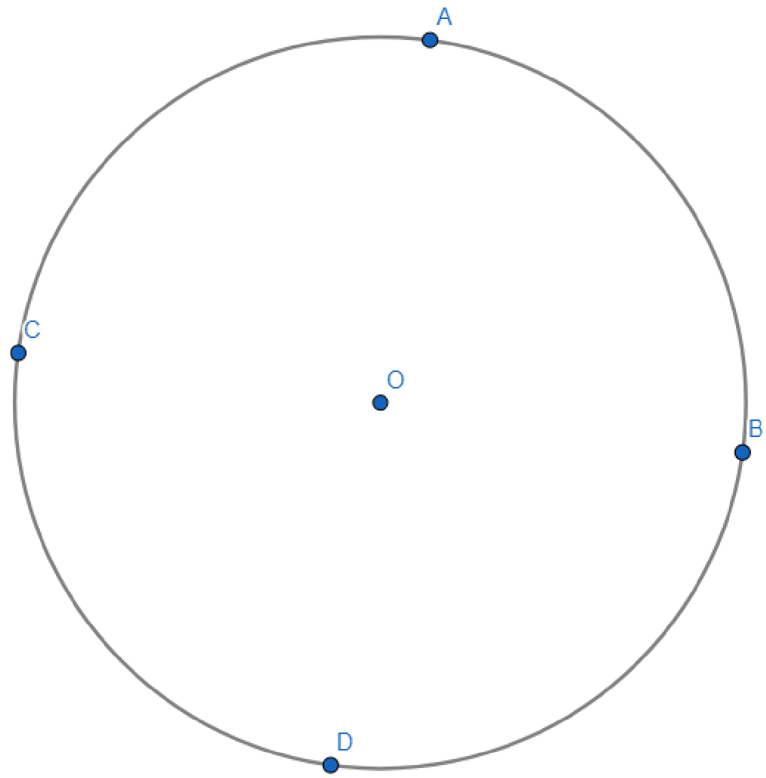


Figure 3

Problem 20. (Challenge problem) Write the full symmetry group for Figure 3 (make sure to include the reflectional symmetry).

5. "Sub"-symmetries and Subgroups

Let $(G, \cdot)$ be a group, and let $F \subset G$ (and $F \neq G$). If $(F, \cdot)$ is a group (ie. it satisfies the Group Axioms), we call F a **proper subgroup** of G. It is true that G is a **subgroup** of G, but it is not a *proper* subgroup.

Theorem. (Lagrange) Let G be a finite group and let F be a subgroup of G. Then F is a finite group and $|F|$ divides $|G|$.

We will not be proving this, but you may use it.

Problem 21. Consider the group $G = \langle a | a^k \rangle$ for some $k > 1$. Describe all the subgroups of G.

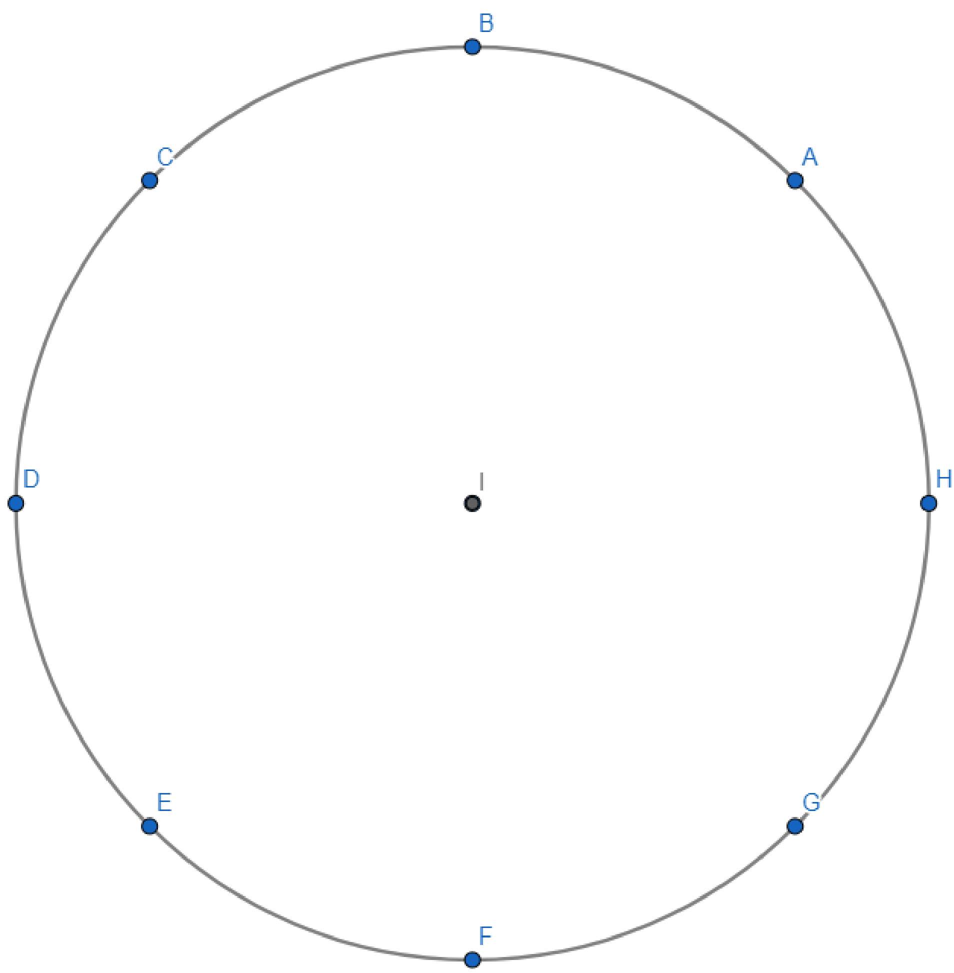


Figure 4

Problem 22. Let G be the rotational symmetry group of Figure 4. What are the subgroups of G?

Problem 23. For each subgroup you found in Problem 22, draw a figure with rotational symmetry group equal to that subgroup.

Problem 24. Generalise the previous three problems. That is, given a positive integer k, draw a polygon with k points, and find the rotational symmetry group G of that polygon. Find, with proof, all figures that have a rotational group equal to a subgroup of G.

6. Challenge Problems

Problem 25. Prove that for any prime p, the set $G = \{1, 2, 3, 4, \ldots, p-1\}$ with multiplication modulo p forms a group.

Problem 26. Let G be a finite group and $a \in G$. Prove there is an integer $0 < k < |G|$ so that $a^k = e$.

Problem 27. Prove Fermat's Little Theorem: for any prime number p and any integer $a > 0$, $a^p \equiv a \mod p$.

Problem 28. Prove Lagrange's Theorem: Let G be a finite group and let F be a subgroup of G. Then F is a finite group and $|F|$ divides $|G|$.

7. Extra Challenging Very Hard Challenge Problems

Problem 29. Show that if K and N are two finite subgroups of a group G of relatively prime orders, then $K \cap N = \{e\}$.

Problem 30. Show that if a group G has only finite number of subgroups, then G is finite.

Problem 31. Show that if $a^2 = e$ for all elements a of a group G, then G is abelian (ie. $ab = ba$ for every $a, b \in G$).

Problem 32. Prove that if G is a finite group of even order, then G contains an element a such that $a^2 = e$ and $a \neq e$.