Number Theory

Sahana Sri

October 15, 2023

1 Warm Up

Three-fourths of a pitcher is filled with pineapple juice. The pitcher is emptied by pouring an equal amount of juice into each of 5 cups. What percent of the total capacity of the pitcher did each cup receive?

(A) 5 (B) 10 (C) 15 (D) 20 (E) 25

If the degree measures of the angles of a triangle are in the ratio $3 : 3 : 4$, what is the degree measure of the largest angle of the triangle?

(A) 18 (B) 36 (C) 60 (D) 72 (E) 90

When Clara totaled her scores, she inadvertently reversed the units digit and the tens digit of one score. By which of the following might her incorrect sum have differed from the correct one?

(A) 45 (B) 46 (C) 47 (D) 48 (E) 49

A number of students from Fibonacci Middle School are taking part in a community service project. The ratio of 8th graders to 6th graders is $5 : 3$, and the the ratio of 8th graders to 7th graders is $8 : 5$. What is the smallest number of students that could be participating in the project?

(A) 16 (B) 40 (C) 55 (D) 79 (E) 89

2 Divisibility

You can easily test if a number can be exactly divided by another number! Any number that is divisible by another number is called *composite*. Any number that is only divisible by 1 and itself is called *prime*.

2.1 Primes

It's time for your favorite activity, the Sieve of Eratosthenes! Using the following 1-100 grid, go through the following procedures to find all the prime numbers less than 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1. Cross out 1 by Shading in the box completely. One is neither prime nor composite.
2. Use a forward Slash to cross out all multiples of 2 (starting with 4 since 2 is prime).
3. Use a backward Slash / to cross out all multiples of 3 (starting with 6 since 3 is prime).
4. Multiples of 4 have been crossed out already (multiples of 2).
5. Draw a Square on all multiples of 5 (starting with 10 since 5 is prime).
6. Multiples of 6 have been crossed out already (multiples of 2 and 3).
7. Circle all multiples of 7 (starting with 14 since 7 is prime).
8. Multiples of 8 have been crossed out already (multiples of 2).
9. Multiples of 9 have been crossed out already (multiples of 3).
10. Multiples of 10 have been crossed out already (multiples of 2 and 5).

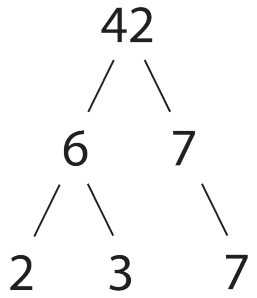
2.2 Composite

Now that we have looked at prime numbers, we can look at composite numbers. Composite numbers are divisible by numbers other than 1 and themselves, which means we can use the following divisibility rules to find what numbers they are divisible by and eventually, their prime factorization.

A number is divisible by:
2: if a number is even (last digit is divisible by 2)
3: if the sum of its digits is divisible by 3
4: if its last two digits are divisible by 4
5: if its last digit is 0 or 5
6: if it is divisible by 2 and 3
7: ...just divide. There's no good trick for this one.
8: if the number formed by the last three digits is divisible by 8. Yes, you may have to just do the division for this.
9: if the sum of digits is divisible by 9
10: if the last digit is 0
11: Calculate the sum of odd positioned digits (O) and even positioned digits (E). If O - E is divisible by 11, then the number is also divisible by 11. Don't forget that O - E can be negative!

Now that we know these divisibility rules, we can find the prime factorizations of numbers, which expresses a number as a product of prime numbers.

According to the Fundamental Theorem of Arithmetic, every number has it's own **unique** prime factorization. The easiest way to find a prime factorization is to use a factor tree.



The above is an example which shows the prime factorization for 42 in a factor tree. Each factor creates a new "branch," and any composite factors are broken down completely into prime factors.

2.3 Types of Questions

There will be many different questions the AMC test might ask where factors will be needed, but here are some of the most common usages.

2.3.1 Number of Factors

The sum of the factors of an integer n, where a, b, c etc. represent the prime factors of n and $(a)^p(b)^q(c)^r...$ is the prime factorization of n, is

$$\frac{a^{p+1}-1}{a-1}\frac{b^{q+1}-1}{b-1}\frac{c^{r+1}-1}{c-1}...$$

Try finding the sum of the factors of the following numbers:
a) 375

b) 360

2.3.2 Product of Factors

The product of the factors of an integer n, where S is the sum of its factors as defined above, is

$$n^{\frac{S}{2}}$$

Try finding the product of the factors of the following numbers:
a) 375

b) 360

3 Practice!

Which of the following numbers has the smallest prime factor?
(A) 55 (B) 57 (C) 58 (D) 59 (E) 61

The 5-digit number 2 0 1 8 U is divisible by 9. What is the remainder when this number is divided by 8?

The 7-digit numbers $74A52B1$ and $326AB4C$ are each multiples of 3. Which of the following could be the value of C?
(A) 1 (B) 2 (C) 3 (D) 5 (E) 8

The sum of two prime numbers is 85. What is the product of these two prime numbers?

What is the sum of the two smallest prime factors of 250?

Call a positive integer an uphill integer if every digit is strictly greater than the previous digit. For example, 1357, 89, and 5 are all uphill integers, but 32, 1240, and 466 are not. How many uphill integers are divisible by 15?

The number 64 has the property that it is divisible by its units digit. How many whole numbers between 10 and 50 have this property?

Ayasha, Beshkno, and Chenoa were all born after 2000. Each of them was born in a year after 2000 that is divisible by exactly one of the prime numbers 2, 3 or 5. Each of these primes is a divisor of one of the birth years. What is the least possible sum of their birth years?

How many permutations of 123456 are divisible by their last digit? For instance, 123456 is divisible by 6, but 561234 is not divisible by 4.

How many positive factors does $23,232$ have?

On June 1, a group of students is standing in rows, with 15 students in each row. On June 2, the same group is standing with all of the students in one long row. On June 3, the same group is standing with just one student in each row. On June 4, the same group is standing with 6 students in each row. This process continues through June 12 with a different number of students per row each day. However, on June 13, they cannot find a new way of organizing the students. What is the smallest possible number of students in the group?

Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

How many factors of 2020 have more than 3 factors? (Example, 12 has 6 factors, namely $1, 2, 3, 4, 6$, and 12.)

A positive integer q is the product of a prime number and a perfect square. Additionally, q is the product of a different prime number and a perfect cube. What is the least possible value of q?

How many positive multiples of 1,001 have less than 15 factors?

Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

How many positive divisors of 2020 do not also divide 1010?

What is the sum of $\frac{1}{a}$ over all positive factors a of the number 360?

Which of the following expressions is never a prime number when p is a prime number?
(A) $p^2 + 16$ (B) $p^2 + 24$ (C) $p^2 + 26$ (D) $p^2 + 46$ (E) $p^2 + 96$

One of the following numbers is not divisible by any prime number less than 10. Which is it?
(A) $2^{606}-1$ (B) $2^{606}+1$ (C) $2^{607}-1$ (D) $2^{607}+1$ (E) $2^{607}+3^{607}$