Olympiads Week 3: Invariants

ORMC

10/8/23

Remember to *write down your solutions, as proofs.* You don't have to start by writing out a full proof to every problem you try, but once you've solved a problem or two, take a few minutes to write out a proof as if this was being graded at an Olympiad.

1 Practice Problems: Invariants

These problems can be solved using *invariants*. When there is some repeated process, rather than studying what does change, we may want to look at what stays the same. This allows us to make connections between the starting and ending positions, and we can rule out many possibilities this way.

We'll work through this one together:

Problem 1.1 (46th IMO)**.** There are n markers, each with one side white and the other side black, aligned in a row with their white sides up. At each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it, and reverse the two neighboring markers. Prove that one can reach a configuration with only two markers left if and only if $n-1$ is not divisible by 3.

Problem 1.2. Suppose we put pawns on opposite corners of a chessboard. Can you then cover the remaining 62 squares of the chessboard with 31 non-overlapping 2×1-square dominoes?

The rest of these problems are from Putnam and Beyond.

Problem 1.3. An ordered triple of numbers is given. It is permitted to perform the following operation on the triple: to change two of them, say a and b, to $(a+b)/\sqrt{2}$ and $(a-b)/\sqrt{2}$. Is it possible to obtain the triple $(1, \sqrt{2}, 1+\sqrt{2})$ from the triple $(2, \sqrt{2}, 1/\sqrt{2})$ using this operation?

Problem 1.4. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones by performing the operation finitely many times?

Problem 1.5. The number $99\ldots99$ (having 1997 nines) is written on a blackboard. Each minute, one number written on the blackboard is factored into two factors and erased, each factor is (independently) increased or decreased by 2, and the resulting two numbers are written. Is it possible that at some point all of the numbers on the blackboard are equal to 9?

Problem 1.6. Four congruent right triangles are given. One can cut one of them along the altitude and repeat the operation several times with the newly obtained triangles. Prove that no matter how we perform the cuts, we can always find among the triangles two that are congruent.

2 Competition Problems

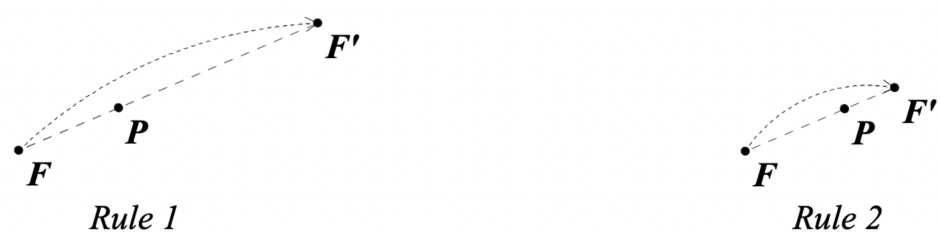
Problem 2.1 (BAMO 2011 Problem 1)**.** Hugo plays a game: he places a chess piece on the top left square of a 20×20 chessboard and makes 10 moves with it. On each of these 10 moves, he moves the piece either one square horizontally (left or right) or one square vertically (up or down). After the last move, he draws an X on the square that the piece occupies. When Hugo plays this game over and over again, what is the largest possible number of squares that could eventually be marked with an X? Prove that your answer is correct.

Problem 2.2 (BAMO 2008 Problem 1)**.** Call a year ultra-even if all of its digits are even. Thus 2000, 2002, 2004, 2006, and 2008 are all ultra-even years. They are all 2 years apart, which is the shortest possible gap. 2009 is not an ultra-even year because of the 9, and 2010 is not an ultra-even year because of the 1.

- In the years between the years 1 and 10000, what is the longest possible gap between two ultra-even years? Give an example of two ultra-even years that far apart with no ultra-even years between them. Justify your answer.
- What is the second-shortest possible gap (that is, the shortest gap longer than 2 years) between two ultra-even years? Again, give an example, and justify your answer.

Problem 2.3 (BAMO 2009 Problem 6)**.** At the start of this problem, six frogs are sitting with one at each of the six vertices of a regular hexagon. Every minute, we choose a frog to jump over another frog using one of the two rules illustrated below. If a frog at point F jumps over a frog at point P, the frog will land at point F' such that F, P, and F' are collinear and

- using Rule 1, $F'P = 2FP$.
- using Rule 2, $F'P = FP/2$.



It is up to us to choose which frog to take the leap and which frog to jump over.

1. If we only use Rule 1, is it possible for some frog to land at the center of the original hexagon after a finite amount of time?
2. If both Rule 1 and Rule 2 are allowed (freely choosing which rule to use, which frog to jump, and which frog it jumps over), is it possible for some frog to land at the center of the original hexagon after a finite amount of time?

Problem 2.4 (BAMO 2012 Problem 6)**.** Given a segment AB in the plane, choose on it a point M different from A and B. Two equilateral triangles ΔAMC and ΔBMD in the plane are constructed on the same side of segment AB. The circumcircles of the two triangles intersect in point M and another point N. (The circumcircle of a triangle is the circle that passes through all three of its vertices.)

1. Prove that lines AD and BC pass through point N.

2. Prove that no matter where one chooses the point M along segment AB, all lines MN will pass through some fixed point K in the plane.

Problem 2.5 (USAMO 2011 Problem 2)**.** An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding 2m to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

Problem 2.6 (USAMO 2015 Problem 4)**.** Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finished piling his stones in some manner, he can then perform stone moves, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i, k), (i, l), (j, k), (j, l)$ for some $1 \leq i, j, k, l \leq n$, such that $i < j$ and $k < l$. A stone move consists of either removing one stone from each of (i, k) and (j, l) and moving them to (i, l) and (j, k) respectively,j or removing one stone from each of (i, l) and (j, k) and moving them to (i, k) and (j, l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

Problem 2.7 (Putnam 2010 Problem B3)**.** There are 2010 boxes labeled $B_1, \dots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from B_i into any other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

Problem 2.8. Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

- places a stone in an empty space, or
- removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?