Intermediate 1 **Oct 15, 2023**

More on Counting

1 Counting Review

Last time, we learned the notion of a factorial:

$$n! = n \times (n-1) \times \cdots \times 1.$$

For example, $3! = 3 \times 2 \times 1 = 6$. As a convention, we set $0! = 1$.

Problem 1 *Compute the following:*

a. $1! =$

b. $2! =$

c. $4! =$

d. $5! =$

e. $\dfrac{5!}{2! \cdot 3!} =$

f. $\dfrac{1000!}{998! \cdot 2!} =$

Problem 2 *How many ways are there to place the letters 'M', 'A', 'T', 'H' together in a row, ignoring whether it's a valid word?*

Recall that a way to choose k objects out of n so that the **order of the chosen objects matters** is called a **permutation**. the number of permutations is given by the following formula.

$$P(n,k) = \frac{n!}{(n-k)!} \tag{1}$$

Problem 3 *Compute the following:*

a. $P(3,0) =$

b. $P(3,1) =$

c. $P(3,2) =$

d. $P(3,3) =$

e. $P(5,4) =$

f. $P(100,1) =$

Problem 4 *There are 25 students in a class. The teacher wants to choose one student from the class to erase the board, and another student to clean the floor. In how many ways can the teacher choose the students?*

Suppose now the order of the chosen things does not matter. A way to choose k objects out of n so that the order of the chosen objects **does not matter** is called a **combination**. The formular is given by

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = C(n,k) \tag{2}$$

Problem 5 *Compute the following:*

$a.\quad \binom{4}{0} =$

$b.\quad \binom{4}{1} =$

$c.\quad \binom{4}{2} =$

$d.\quad \binom{4}{3} =$

$e.\quad \binom{4}{4} =$

$f.\quad \binom{100}{1} =$

Problem 6 *There are 25 students in a class. The teacher wants to choose 2 students from the class to erase the board. In how many ways can the teacher choose the students?*

What is the difference between Problem 4 and 6?

2 Let's Flip Coins

Form groups of 4 or 5 people. Toss 4 coins one time each and record how many heads you get in the space below. Repeat this 20 times, and make sure you write down the number of heads each time.

Trial #	The Number of Heads	Trial #	The Number of Heads
1		11	
2		12	
3		13	
4		14	
5		15	
6		16	
7		17	
8		18	
9		19	
10		20	

Problem 7 *Out of the 20 trials you did, what is the percentage of times that you got the following outcomes?*

0 heads:

1 head:

2 heads:

3 heads:

4 heads:

Problem 8 *We toss 4 coins at the same time. List all possible ways to get the following outcomes. Use H to represent heads and T to represent tail*

0 heads: TTTT.

1 head:

2 heads:

3 heads:

4 heads:

Problem 9 *What is the total number of outcomes you can get by tossing 4 coins. (Order matters!!) Calculate this using math, then verify it by counting all possible outcomes in the previous problem.*

Problem 10 *When tossing 4 coins, what is the probability of the following outcomes? Calculate using combination formula, then verify by counting the results in Problem 8.*

0 heads:

1 head:

2 heads:

3 heads:

4 heads:

Compare your answer to Problem 7 and 10. Is the result from your experiment in Problem 7 the same or different from your computation in Problem 10?

Problem 11 *You toss a fair coin 10 times. What is the chance that you get 5 tails?*

Problem 12 *You toss a fair coin 16 times. (Hint:* $2^{16} = 65536$*)*

- *What is the chance that you get no tails?*

- *What is the chance that you get strictly less than 3 tails?*

- *What is the chance that you get at least 3 tails?*

3 More Probability Problems

Problem 13 *A club has 10 members, 5 boys and 5 girls. Two of the members are chosen at random. What is the probability that they are both girls?*

Problem 14 *Another club has 20 members, 12 boys and 8 girls. Two of the members are chosen at random. What is the probability that a boy and a girl are chosen?*

Problem 15 *What is the probability that a random arrangement of the letters in the word 'SEVEN' will have both E's next to each other?*

Problem 16 *2 vertices of an octagon are chosen at random. What is the probability that they are adjacent?*

Problem 17 *A fair coin is tossed 4 times. What is the chance of getting more HEADS than TAILS?*

Problem 18 *Throw 3 dices at the same time. What is the probability that exactly two out of the three dices turn out to be number 6.*

The probability of X happens k times out of n independent trials is known as **binomial distribution**. Assume the probability of X happening is p. If we repeatedly doing the same thing for n times, the probability of X happening for exactly k times is

$$P(X \text{ happens for } k \text{ times}) = \binom{n}{k} p^k (1-p)^{n-k}.$$

For example, we know that when throwing a 6-sided dice, the probability of geting a 1 is $\frac{1}{6}$. If we throw the dice for 5 times, the probability of getting 1 exactly 3 times is

$$\binom{5}{3}\left(\frac{1}{6}\right)^3\left(1-\frac{1}{6}\right)^{5-3} = \frac{250}{7776}.$$

Suppose that the coin you toss is not necessarily fair. At every toss, the chance to get a head is $\frac{2}{3}$.

Problem 19 *What is the chance to get a tail?*

Problem 20 *You toss the above coin 7 times. What is the chance that you get 3 heads?*

Problem 21 *You toss the above coin 7 times. What is the chance that you get no more than 3 heads?*

Problem 22 *Hospital records show that of patients suffering from a certain disease, 75% die of it. What is the probability that of 5 randomly selected patients, exactly 4 will recover?*

Problem 23 *When Oleg calls his daughter Anya, on vacation the Russian village of Borodulino, the chance of the call getting through is 60%. How likely is it to have at least one connection in four calls?*

Problem 24 *The chance of a runner to improve his own personal record in a race is p. What is the chance that he improves his record in at most three races?*

Problem 25 *You toss a pair of fair dice five times. What is the chance that you get ten two times?*

Problem 26 *You toss a pair of fair dice five times. What is the chance that you get ten at least two times?*

Problem 27 *A pharmaceutical study shows that a new drug causes negative side effects in 3 of every 100 patients. To check the number, a different lab chooses 5 random people to take the drug. What is the likelihood of the following events?*

1. *None of the five patients experiences side effects.*

2. *At least two experience side effects.*

Problem 28 *You pick up a natural (positive integral) number at random. What is the chance that the number is divisible either by two or by three?*

Problem 29 *Three players are tossing a fair coin. The first to have a HEAD wins. What are the players' chances of winning?*

Problem 30 *3 cards are chosen at random from a standard 52-card deck. What is the probability that they form a pair? (A 3-card hand is a "pair" if two of the cards match in rank but the third card is different. For example, 668 is a pair, but 999 is not.)*

Problem 31 *Perhaps the most common error in ranking poker hands involves juxtaposing the order of three of a kind and two pairs. Compare the probability of getting three of a kind (e.g. 22245) with that of getting two pairs (e.g. 22445). Can you offer an intuitive argument that would have allowed you to determine which is more likely without performing any calculation at all?*

Problem 32 *2 diagonals of a regular heptagon (a 7-sided polygon) are chosen. What is the probability that they intersect inside the heptagon?*

Problem 33 *I have 120 blocks. Each block is one of 2 different materials, 3 different colors, 4 different sizes, and 5 different shapes. No two blocks have exactly the same of all four properties. I take two blocks at random. What is the probability the two blocks have exactly two of these four properties the same?*