ORMC AMC 10/12 Group
Week 3: Complex Numbers

October 15, 2023

1 Warm-up Exercises

1. Write $\frac{1}{3+2i}$ in the form $a + bi$, where $i = \sqrt{-1}$.

2. Factor $x^4 + 1$ into polynomials with real coefficients.

3. Solve $3z + 4\overline{z} = 12 - 5i$ for z.

4. Let $S = in + i^{-n}$, where n is an integer and $i = \sqrt{-1}$. Find the total number of possible distinct values of S.

5. Let $i = \sqrt{-1}$. Find

$$\cfrac{1}{1 + \cfrac{1}{1 - \cfrac{1}{1+i}}}$$

6. If $i = \sqrt{-1}$, evaluate $i + 2i^2 + 3i^3 + 4i^4 + \cdots + 64i^{64}$

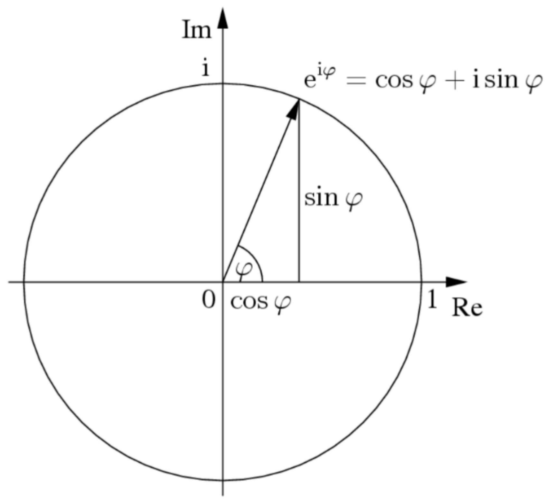
2 Complex Number Basics

As you may already know, the complex numbers are what we get when we take the real numbers and add $i = \sqrt{-1}$. Every complex number is of the form $z = a + bi$, where a and b are real numbers. We call a the **real part** of the number (written $\Re(z) = a$), and b the **imaginary part** (written $\Im(z) = b$).

One important operation to know is **complex conjugation**, which means negating the imaginary part of the complex number. It is denoted as $\overline{z}$. So, if $z = a + bi$, then $\overline{z} = a - bi$. Another important property of complex numbers is the **modulus**, or the "length" of the complex number. This is similar to euclidean distance, and for $z = a + bi$, it is $|z| = \sqrt{a^2 + b^2}$. Notice also that $|z|^2 = z \cdot \overline{z}$. This is a very useful identity to remember, since it is often useful to be able to change absolute value into basic multiplication.

We generally graph complex numbers in the cartesian plane, using the x-axis for the real part of the number, and the y-axis for the imaginary part of the number.

Just like how we have polar coordinates when graphing 2-D real numbers, we also have a polar form for complex numbers. As usual, we require a "radius", which in this case is the modulus, and an "angle", which in this case is called the **argument**. The argument is the angle the complex number makes with the positive real x-axis. For example, the argument in the diagram below is φ.



Notice that the diagram above also shows the conversion from polar to cartesian representation of complex numbers. Given the modulus r and argument φ of a complex number, its real part is $r\cos(\varphi)$ and its imaginary part is $r\sin(\varphi)$. We will come back to the $e^{i\varphi}$ portion in next week's worksheet.

3 Examples

1. Find all complex numbers z for which $\frac{2z-3i}{z+4} = -5 + i$

2. Find the set of all complex numbers z such that $z/\overline{z}$ is a real number, and the set for which $z/\overline{z}$ is pure-imaginary.

3. Let z be a complex number that satisfies $z + 6i = iz$. Find z.

4. Find the area of the region enclosed by the graph of $|z - 4 + 5i| = 2\sqrt{3}$.

4 Exercises

1. **(1991 AHSME #18)** Find the set of points z in the complex plane such that $(3+4i)z$ is a real number.

2. **(1984 AHSME #10)** Four complex numbers lie at the vertices of a square in the complex plane. Three of the numbers are $1+2i, -2+i$, and $-1-2i$. Find the fourth number.

3. **(2009 AIME I #2)** There is a complex number z with imaginary part 164 and a positive integer n such that
$$\frac{z}{z+n} = 4i.$$
Find n.

4. **(2009 AMC 12A #15)** For what value of n is $i + 2i^2 + 3i^3 + \cdots + ni^n = 48 + 49i$?

5. **(2004 AMC 12B #16)** A function f is defined by $f(z) = i\overline{z}$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of z. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

6. **(1985 AIME #3)** Find c if a, b, and c are positive integers which satisfy $c = (a+bi)^3 - 107i$, where $i^2 = -1$.

7. **(2007 AMC 12 #18)** The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2+i) = 0$. What is $a+b+c+d$?

8. **(2019 AMC 12A #14)** For a certain complex number c, the polynomial

$$P(x) = (x^2 - 2x + 2)(x^2 - cx + 4)(x^2 - 4x + 8)$$

has exactly 4 distinct roots. What is $|c|$?

9. **(1995 AIME #5)** For certain real values of a, b, c, and d, the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ has four non-real roots. The product of two of these roots is $13+i$ and the sum of the other two roots is $3+4i$, where $i = \sqrt{-1}$. Find b.

10. **(2018 AMC 12A #22)** The solutions to the equations $z^2 = 4 + 4\sqrt{15}i$ and $z^2 = 2 + 2\sqrt{3}i$, where $i = \sqrt{-1}$, form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form $p\sqrt{q} - r\sqrt{s}$, where p, q, r, and s are positive integers and neither q nor s is divisible by the square of any prime number. What is $p+q+r+s$?

11. **(1994 AIME #8)** The points $(0,0)$, $(a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of ab.

12. Two solutions of $x^4 - 3x^3 + 5x^2 - 27x - 36 = 0$ are pure imaginary numbers. Find these two solutions.

13. **(2009 AMC 12B #23)** A region S in the complex plane is defined by

$$S = \{x + iy : -1 \leq x \leq 1, -1 \leq y \leq 1\}.$$

A complex number $z = x + iy$ is chosen uniformly at random from S. What is the probability that $\left(\frac{3}{4} + \frac{3}{4}i\right) z$ is also in S?

14. **(2008 AMC 12B #19)** A function f is defined by $f(z) = (4+i)z^2 + \alpha z + \gamma$ for all complex numbers z, where α and γ are complex numbers and $i^2 = -1$. Suppose that $f(1)$ and $f(i)$ are both real. What is the smallest possible value of $|\alpha| + |\gamma|$?

15. **(1992 AIME #10)** Consider the region A in the complex plane that consists of all points z such that both $\frac{z}{40}$ and $\frac{40}{\overline{z}}$ have real and imaginary parts between 0 and 1, inclusive. What is the integer that is nearest the area of A?

16. **(2011 AMC 12A #23)** Let $f(z) = \frac{z+a}{z+b}$ and $g(z) = f(f(z))$, where a and b are complex numbers. Suppose that $|a| = 1$ and $g(g(z)) = z$ for all z for which $g(g(z))$ is defined. What is the difference between the largest and smallest possible values of $|b|$?

17. **(2009 AMC 12A #21)** Let $p(x) = x^3 + ax^2 + bx + c$, where a, b, and c are complex numbers. Suppose that

$$p(2009 + 9002\pi i) = p(2009) = p(9002) = 0$$

What is the number of nonreal zeros of $x^{12} + ax^8 + bx^4 + c$?

18. **(2004 AMC 12A #23)** A polynomial

$$P(x) = c_{2004}x^{2004} + c_{2003}x^{2003} + ... + c_1 x + c_0$$

has real coefficients with $c_{2004} \neq 0$ and 2004 distinct complex zeroes $z_k = a_k + b_k i$, $1 \leq k \leq 2004$ with a_k and b_k real, $a_1 = b_1 = 0$, and

$$\sum_{k=1}^{2004} a_k = \sum_{k=1}^{2004} b_k.$$

Which of the following quantities can be a non zero number?

(A)c_0 (B)c_{2003} (C)$b_2 b_3 ... b_{2004}$ (D) $\sum_{k=1}^{2004} a_k$ (E) $\sum_{k=1}^{2004} c_k$

19. Show that the equation

$$\frac{|z-2|}{|z+1|} = 2$$

makes a circle in the complex plane, by converting it to the form $|z - c| = r$. What if we replace z by $1/z$?

20. Let u be a complex number, and let v be the reflection of u over the line $\Re(z) = \Im(z)$. Express v in terms of u and $\overline{u}$.