More Induction

Compiled by Andreas and Nakul

1 Introduction

Last time, we introduced induction and solved some basic examples. We will continue this week with more problems. First, let's go over some more questions similar to those we have already seen.

Problem 1.1.
Prove that $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$ for all $n \in \mathbb{N}$.

Solution. We will solve the first one for you so you get an idea of how to proceed. Before diving into the actual proof, it's a good idea to convince yourself that the statement really is true. We start by verifying it for a few base cases:

$$1 \cdot 1! = 1 = 2! - 1$$

$$1 \cdot 1! + 2 \cdot 2! = 5 = 3! - 1$$

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! = 15 = 4! - 1$$

The last calculation could be made slightly simpler by observing, from the second last step, that $1 \cdot 1! + 2 \cdot 2! = 3! - 1$. Using this, we redo the last case:

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! = 3! - 1 + 3 \cdot 3! = (1+3) \cdot 3! - 1 = 4! - 1$$

Using this method, we never explicitly find the value 15. But this is a lot easier to generalize. For the next step, we write

$$\sum_{i=1}^{4} i \cdot i! = \sum_{i=1}^{3} i \cdot i! + 4 \cdot 4! = 4! - 1 + 4 \cdot 4! = 5! - 1$$

At this point, you might have noticed that if you know that the formula is true for a certain $k \in \mathbb{N}$, it is not too hard to show that it is also true for $k+1$. This is exactly what induction requires! Now, let's formally write out the argument that we have been hinting at.

Base Case:
We'll check the statement for $n = 1$:

$$1 \cdot 1! = 1$$

and

$$(1+1)! - 1 = 2! - 1 = 2 - 1 = 1$$

Since the left-hand side (LHS) equals the right-hand side (RHS), the base case holds.

Inductive Step:
Assume the statement is true for some arbitrary positive integer k:

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k! = (k+1)! - 1 \tag{1}$$

Now, we'll prove the statement for $n = k + 1$:

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k! + (k+1) \cdot (k+1)!$$

To prove that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k! + (k+1) \cdot (k+1)! = (k+2)! - 1$$

We can use the assumption from equation (1) to substitute the sum from 1 to k:

$$(k+1)! - 1 + (k+1) \cdot (k+1)! = (k+2)! - 1$$

After simplifying the expression, the left-hand side (LHS) equals the right-hand side (RHS). Therefore, the expression

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + (k+1) \cdot (k+1)! = (k+2)! - 1$$

holds true when the expression

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k! = (k+1)! - 1$$

is assumed to be true. Thus, by the principle of mathematical induction, the statement

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$$

is true for all $n \in \mathbb{N}$.

□

Problem 1.2.
Prove that $\sum_{r=1}^{n} r(r+1) = \dfrac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{N}$.

Problem 1.3.
Prove that $\sum_{i=1}^{n} i \cdot 2^i = (n-1)2^{n+1} + 2$ for all $n \in \mathbb{N}$.

Problem 1.4.
Prove that $\sum_{j=1}^{n} (j+1)2^{j-1} = n2^n$ for all $n \in \mathbb{N}$.

Problem 1.5.
Prove that $\dfrac{1}{1 \times 2} + \dfrac{1}{2 \times 3} + \dfrac{1}{3 \times 4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$ for all $n \in \mathbb{N}$.

2 Inequalities

In most of the problems you have seen so far, the principle of induction is being used to prove a certain equality. However, we can use the same idea to prove an *inequality.* Here are a few examples. Again, we've solved the first one for you.

Problem 2.1.
Prove that $n! > 2^n$ for all $n \geq 4$.

Solution. This time, we will leave it up to you to convince yourself this is true by inspecting some base cases.
Inductive Step:
Assume that the inequality holds true for some arbitrary positive integer k such that

$$k! > 2^k$$

Now, we'll prove the inequality for $n = k + 1$:

$$(k + 1)! > 2^{k+1}$$

We know that

$$(k + 1)! = (k + 1) \cdot k!$$

Using the inductive hypothesis, we substitute the assumed inequality for $k!$:

$$(k + 1)! = (k + 1) \cdot k! > (k + 1) \cdot 2^k$$

To prove the inductive step, we need to show that

$$(k + 1) \cdot 2^k > 2^{k+1}$$

Simplifying the expression, we observe that for $k \geq 4$, the expression holds true, thus confirming the inductive step. Therefore, by the principle of mathematical induction, the inequality

$$n! > 2^n$$

holds true for all $n \geq 4$.

$\square$

Problem 2.2.
Prove that $4^n < (n + 4)!$ for all $n \in \mathbb{N}$.

Problem 2.3.
Prove that $(1 + 1/n)^n < 3$ for all $n \geq 1$.

Problem 2.4.
Prove by induction that if p is any real number satisfying $p > -1$, then

$$(1 + p)^n \geq 1 + np$$

for all $n \in \mathbb{N}$.

Problem 2.5.
The Fibonacci sequence f_n consists of a series of numbers starting with $f_1 = 1$, $f_2 = 1$, where for $k > 2$: $f_{k+1} = f_{k-1} + f_k$. In other words, each number is the sum of the two preceeding ones. So, $f_3 = 1 + 1 = 2$, $f_4 = 1 + 2 = 3$, and so on.
Show that for $\phi = \dfrac{1 + \sqrt{5}}{2} \approx 1.62$, satisfying $\phi^2 = \phi + 1$, we have $f_n \geq \phi^{n-2}$.

3 Divisibility

Induction is capable of proving statements that are more general than the "LHS = RHS" or "LHS $\leq$ RHS" type that you have seen so far. In this section, we ask you to prove that certain formulas only produce numbers that have some common divisor.

Problem 3.1.
Prove that $3^{2n} - 1$ is divisible by 8 for all positive integers n.

Solution. **Base Case:**
Let's check the base case for $n = 1$:

$$3^{2\times 1} - 1 = 3^2 - 1 = 8$$

Since 8 is clearly divisible by 8, the base case holds true.
Inductive Step:
Assume that the statement holds for some arbitrary positive integer k such that

$$3^{2k} - 1 \text{ is divisible by } 8$$

Now, we'll prove that the statement holds for $n = k + 1$:

$$3^{2(k+1)} - 1 = 3^{2k+2} - 1$$

Using properties of exponents, we can rewrite the expression as:

$$(3^{2k})(3^2) - 1$$

We know from the inductive hypothesis that $3^{2k} - 1$ is divisible by 8. Let's denote this as

$$3^{2k} - 1 = 8m$$

for some integer m. Substituting back into the expression, we get

$$3^{2k+2} - 1 = (3^{2k})(3^2) - 1$$

$$= 9 \times (8m + 1) - 1$$
$$= 72m + 9 - 1$$
$$= 72m + 8$$

Since $72m$ is clearly divisible by 8 and 8 is also divisible by 8, the expression

$$72m + 8$$

is divisible by 8. Thus, if the statement holds for an arbitrary positive integer k, it also holds for $k + 1$. Therefore, by the principle of mathematical induction, the statement

$$3^{2n} - 1 \text{ is divisible by } 8$$

holds for all positive integers n. □

Problem 3.2.
Prove that $9^n - 2^n$ is divisible by 7 for all positive integers n.

Problem 3.3.
Prove that $5^n - 1$ is divisible by 4 for all positive integers n.

Problem 3.4.
Prove that $11^n - 6^n$ is divisible by 5 for all positive integers n.

Problem 3.5.
Do you notice a pattern? Prove that for $x, y \in \mathbb{N}$, $x^n - y^n$ is divisible by $x - y$ for all positive integers n.

4 Some visual problems

If you have not yet been convinced of the strength of induction, here are some examples that will require you to draw pictures and use your imagination.

Problem 4.1.
Prove that a $2 \times n$ board can be completely covered by n dominoes (each of size 2×1 or 1×2) for all $n \in \mathbb{N}$.

Solution. As usual, we will use mathematical induction.
Base Case:
For $n = 1$, the 2×1 board can certainly be covered by a single 2×1 domino. Thus, the base case holds true.
Inductive Step:
Assume that the statement holds for an arbitrary positive integer k, i.e., a $2 \times k$ board can be covered by k dominoes.

Now, we need to prove that the statement holds for $n = k + 1$. Consider a $2 \times (k + 1)$ board. According to our inductive hypothesis, a $2 \times k$ board can be covered by k dominoes.

Next, place a domino on the $2 \times k$ board to cover the two remaining squares (the ones in column $k+1$). The domino will cover these two squares perfectly, and we have covered the entire $2 \times (k + 1)$ board using $k + 1$ dominoes.

Thus, if the statement holds for some arbitrary positive integer k, it also holds for $k + 1$. Therefore, by the principle of mathematical induction, the statement holds for all $n \in \mathbb{N}$.

This inductive proof demonstrates that a $2 \times n$ board can indeed be covered by n dominoes for all natural numbers n.

□

Problem 4.2.
Prove that a $2^n \times 2^n$ chessboard with one square removed can be completely tiled with L-trominoes (an L-tromino is a shape that covers three squares in an L-shape).

Problem 4.3.
Prove that with n cuts, you can create at most 2^n pieces of pizza.

Problem 4.4.
Prove that a rectangular chocolate bar of $m \times n$ squares can be split into mn squares with $mn - 1$ breaks.

Problem 4.5.
Draw n lines in the plane in such a way that no two are parallel and no three intersect in a common point. Prove that the plane is divided into exactly $\frac{n(n+1)}{2} + 1$ parts by the lines.

Problem 4.6.
Draw N circles of varying radii on a piece of paper (intersecting or not). Prove that the resulting map is colorable using only 2 colors. Colorable means that it can be colored such that two neighboring areas don't have the same color.

Problem 4.7.
Prove that you can divide an n-gon, convex or concave, into $n - 2$ triangles ($n > 2$), by drawing diagonals (i.e. connecting verticies). Conclude that the sum of the interior angles of an n-gon is $(n - 2) \cdot 180$.

5 Games

Problem 5.1.
Ask a classmate or an instructor to play this game with you: Let $n \geq 2$ be a natural number. Both of you write a sequence of 0s and 1s according to the following rules. The players start with an empty line and alternate moves. In each move, a player writes 0 or 1 to the end of the current sequence. A player loses if his digit completes a block of n consecutive digits that repeats itself in the sequence for the second time (the two occurrences of the block may overlap). For instance, for $n = 4$, a sequence produced by such a game may look as follows: 00100001101011110011 (the second player lost by the last move because 0011 is repeated).

1. Prove that the game always finishes after finitely many steps.

2. Suppose that n is odd. Prove that the second player (the one who makes the second move) has a winning strategy. Play as the second player against an instructor and see if this strategy works.

3. Show that for $n = 4$, the first player has a winning strategy. Again, test whether your strategy works.

Problem 5.2.
Again, find a partner to play this game: Two players alternate turns to take coins from a pile initially containing n coins. On their turn, a player must take either 1, 2, or 3 coins. The player to take the last coin wins. Prove the following assertions using mathematical induction.

1. If n is congruent to 1, 2, or 3 modulo 4 (i.e., n leaves a remainder of $1, 2,$ or 3 when divided by 4), then the next player to move can always win (assuming optimal play from both players).

2. If n is a multiple of 4, then the player who just moved (not the next player to move) can always win (again, assuming optimal play).

Consider providing an example move sequence for a small n to illustrate the strategy and to validate your proof. Think carefully about the implications of the previous moves on the current position and how a player can force a win.