Selected Solutions for Metrics and Distances

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In this worksheet, we will be exploring the many unexpected ways that distances can be defined. The notion of a space with a distance is called a metric space, and this comes up both in higher-level mathematics (such as analysis, i.e. proof-based calculus) and many real-world applications like in physics and engineering.

1 Introduction

We will begin by solving a few exercises relating to the common notion of distances. Note that $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^n$ denotes the set of ordered lists of elements of $\mathbb{R}$ with n elements.

Problem 1 Calculate the distances between the following points:

(a) -3 and 5 in $\mathbb{R}$
Answer: 8

(b) $(3, -2)$ and $(-5, 7)$ in $\mathbb{R}^2$
Answer: $\sqrt{145}$

(c) $(1, -5, 2)$ and $(-3, 8, 4)$ in $\mathbb{R}^3$
Answer: $3\sqrt{21}$

Problem 2 If $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ are points in $\mathbb{R}^n$, write out the formula for computing the distance.
Solution: $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$

Problem 3 *(Bonus)* Suppose that a square in $\mathbb{R}^2$ has vertices that have x-coordinates of $2, 4, 7$, and 9. What is the area of the square?
Answer: 29

2 Definitions

Informally, a *metric* on a given set is a function that takes two elements of the set and returns the "distance" between the two elements.

Definition 1 If X be a nonempty set and the domain of d is the set of ordered pairs of elements of X, and d is a function that satisfies the following properties:

(i) $d(x, y) \geq 0$ for all x, y in X,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) = d(y, x)$,

(iv) $d(x, z) \leq d(x, y) + d(y, z)$,

then d is called a **metric**. Conditions (i), (iii), and (iv) are known as the **positivity**, **symmetry**, and **triangle inequality** conditions, respectively.

Definition 2 A **metric space** (X, d) is a pair that consists of the set of points X and the metric d.

For our purposes, the set X we are mostly concerned with is $\mathbb{R}^n$.

Definition 3 The **absolute value** of a real number x is defined as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

We will now prove inequalities that will be useful in the later sections of this worksheet.

Problem 4 Show that for any real number a, $-|a| \leq a \leq |a|$.
Sketch: Prove each inequality case-by-case (4 cases)

Problem 5 Show that for any real numbers a and b, we have that $|a + b| \leq |a| + |b|$.
Sketch: Use Problem 4

We will now look at some examples that will use the above definitions and results.

3 Examples of Metrics

Example 1 *(The Metric on $\mathbb{R}$)*

$$d(x, y) := |x - y|$$

for all $x, y \in \mathbb{R}$.

This is also known as the **standard metric** or **standard distance**. The number $d(x, y)$ is the distance between two points on a real number line.

Problem 6 Verify that the function presented in Example 1 is indeed a metric. *Hint: Check if the function d satisfies the conditions in Definition 1.*

Proof of Triangle Inequality:
WTS: $|x - y| \leq |x - z| + |z - y|$. Set $a = x - z$ and $b = z - y$ from Problem 5.

Example 2 *(Euclidean Metric on $\mathbb{R}^2$)*

$$d(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

for $x = (x_1, x_2), y = (y_1, y_2)$.

This is the metric that corresponds to the "distance" we use in our everyday language and the distance that we use to solve problems involving Euclidean geometry.

Lemma 1 *(Cauchy-Schwarz Inequality)* Let x_1, x_2 and y_1, y_2 be real numbers. Then

$$|x_1 y_1 + x_2 y_2| \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}.$$

Problem 7 *(Bonus)* Prove Lemma 1.

Proof:

$$(x_2 y_1 - x_1 y_2)^2 \geq 0$$

$$x_2^2 y_1^2 - 2x_1 x_2 y_1 y_2 + x_1^2 y_2^2 \geq 0$$

$$x_1^2 y_1^2 + 2x_1 x_2 y_1 y_2 + x_2^2 y_2^2 \leq x_1^2 y_1^2 + x_2^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_2^2$$

$$|x_1 y_1 + x_2 y_2| \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

Problem 8 Use the Cauchy-Schwarz inequality to prove that

$$(x_1 + y_1)^2 + (x_2 + y_2)^2 \leq \left(\sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}\right)^2,$$

where x_1, x_2, y_1, and y_2 are any real numbers.

Problem 9 Prove that the function presented in Example 2 is a metric. *Hint: Use the result from Problem 8 to prove the triangle inequality.*

Example 3 *(The Taxicab Metric on $\mathbb{R}^2$)*

$$d_{tc}(x, y) := |x_1 - y_1| + |x_2 - y_2|$$

for $x = (x_1, x_2), y = (y_1, y_2)$.

Problem 10 Prove that the function presented in Example 3 is a metric. For the triangle inequality condition, use the inequality from Problem 5.

Problem 11 Another name for this metric is the Manhattan metric. Manhattan is a part of New York City where the streets form a grid; from the point of view of a bird, the streets of Manhattan look a lot like graph paper! Explain why either Taxicab or Manhattan are applicable names for this metric. Geometrically, how does this differ from the Euclidean metric?

We will now look at applications to urban geography using the Taxicab metric.

Definition 3 Given a set of points $A, B, C, \ldots$ the point or points P at which

$$d_{tc}(P, A) + d_{tc}(P, B) + d_{tc}(P, C) + \ldots$$

is as small as possible will be called the **minimizing region** of the set.

Problem 12 Locate the minimizing region for the following sets:

(a) $A = (-2, 3), B = (1, -4)$.
Solution: Points enclosed by rectangle with A and B as opposite vertices.

(b) $A = (-3, 4), B = (4, 3), C = (0, -2)$.
Solution: (0,3).

Problem 13 Alice, Bruno, and Clyde have to walk to $A = (-3, -1)$, $B = (3, 3)$, and $C = (0, -3)$, respectively. Where should they live so that the sum of the distances they have to walk is a minimum?
Solution: (0,-1).

Problem 14 If a set consists of an odd number of points, what can be said of its minimizing region?

Solution: There is only one point in its minimizing region.

Problem 15 Burger Baron has hamburger stands at $(-5,5)$, $(-2,4)$, $(1,1)$, $(2,6)$, $(5,-2)$, $(3,-4)$, $(-2,-2)$, and $(-4,-4)$. He wants to build a central supply warehouse so that the sum of the distances to the eight hamburger stands is minimized. Where should the warehouse be located?

Solution: One of (-1,0), (0,0), (-1,1), or (0,1).

Problem 16 The Burger Baron stand at $(2,6)$ does a booming business, requiring twice as many deliveries from central supply as each of the other stands. Bearing this in mind, where should the warehouse be located?

Solution: One of ((0,1) or (1,2).

Now, we will work with the concept of an *open ball*, which is a useful notion that leads to the development of topology.

Definition 4 Let (X,d) be a metric space, x be a point in X, and $r>0$. An **open ball** $B_{(X,d)}(x,r)$ centered at x with radius r represents the set of points y such that $d(x,y)<r$. For simplicity, we will just refer to the set of open balls as $B(x,r)$, since it should be clear which metric space we are working in.

Problem 17 Describe and sketch what an open ball of arbitrary radius r represents geometrically in the:

(a) Standard metric on $\mathbb{R}$.

Answer: Open interval on $\mathbb{R}$

(b) Euclidean Metric on $\mathbb{R}^2$.
Answer: Circle

(c) Taxicab Metric on $\mathbb{R}^2$.
Answer: Squares with diagonals on the parallel to the coordinate axes.

Problem 18 Determine which of the following functions are metrics. If the given function is not a metric, provide a counterexample.

(a)

$$d_1(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \qquad x, y \in \mathbb{R}$$

(b)

$$d_2(x,y) = (x-y)^2 \qquad x, y \in \mathbb{R}$$

Note: Does not satisfy the triangle inequality

(c)

$$d_3(a,b) = \max_i |a_i - b_i| \qquad a, b \in \mathbb{R}^n$$

(c)

$$d_4(x, y) = |sin(x) - sin(y)| \quad x, y \in \mathbb{R}$$

Counterexample: $(2\pi, 0)$ *gives a distance of 0.*

Problem 19 For the metrics determined from Problem 18, sketch the open balls of arbitrary radius r.

Note: Only (a) and (c) from 18 are metrics.

Here is an AIME problem that uses the Euclidean metric in $\mathbb{R}^2$.

Problem 20 *(Bonus)* Let $R = (8, 6)$. The lines whose equations are $8y = 15x$ and $10y = 3x$ contain points P and Q, respectively, such that R is the midpoint of $\overline{PQ}$. Find the length of PQ.

Answer: $\frac{60}{7}$