# Metrics and Distances

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In this worksheet, we will be exploring the many unexpected ways that distances can be defined. The notion of a space with a distance is called a metric space, and this comes up both in higher-level mathematics (such as analysis, i.e. proof-based calculus) and many real-world applications like in physics and engineering.

#### 1 Introduction

We will begin by solving a few exercises relating to the common notion of distances. Note that  $\mathbb{R}$  denotes the set of real numbers and  $\mathbb{R}^n$  denotes the set of ordered lists of elements of  $\mathbb{R}$  with n elements.

**Problem 1** Calculate the distances between the following points:

- (a) -3 and 5 in  $\mathbb{R}$
- (b) (3, -2) and (-5, 7) in  $\mathbb{R}^2$
- (c) (1, -5, 2) and (-3, 8, 4) in  $\mathbb{R}^3$

**Problem 2** If  $(x_1,...,x_n)$  and  $(y_1,...,y_n)$  are points in  $\mathbb{R}^n$ , write out the formula for computing the distance.

**Problem 3** (Bonus) Suppose that a square in  $\mathbb{R}^2$  has vertices that have x-coordinates of 2, 4, 7, and 9. What is the area of the square?

### 2 Definitions

Informally, a *metric* on a given set is a function that takes two elements of the set and returns the "distance" between the two elements.

**Definition 1** If X be a nonempty set and the domain of d is the set of ordered pairs of elements of X, and d is a function that satisfies the following properties:

- (i)  $d(x,y) \ge 0$  for all x, y in X,
- (ii) d(x, y) = 0 if and only if x = y,
- (iii) d(x, y) = d(y, x),
- (iv)  $d(x, z) \le d(x, y) + d(y, z)$ ,

then d is called a **metric**. Conditions (i), (iii), and (iv) are known as the **positivity**, **symmetry**, and **triangle inequality** conditions, respectively.

**Definition 2** A metric space (X,d) is a pair that consists of the set of points X and the metric d.

For our purposes, the set X we are mostly concerned with is  $\mathbb{R}^n$ .

**Definition 3** The absolute value of a real number x is defined as

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0. \end{cases}$$

We will now prove inequalities that will be useful in the later sections of this worksheet.

**Problem 4** Show that for any real number  $a, -|a| \le a \le |a|$ .

**Problem 5** Show that for any real numbers a and b, we have that  $|a+b| \leq |a| + |b|$ .

We will now look at some examples that will use the above definitions and results.

# 3 Examples of Metrics

Example 1 (The Metric on  $\mathbb{R}$ )

$$d(x,y) := |x - y|$$

for all  $x, y \in \mathbb{R}$ .

This is also known as the **standard metric** or **standard distance**. The number d(x, y) is the distance between two points on a real number line.

**Problem 6** Verify that the function presented in Example 1 is indeed a metric. *Hint: Check if the function d satisfies the conditions in Definition 1.* 

Example 2 (Euclidean Metric on  $\mathbb{R}^2$ )

$$d(x,y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

for 
$$x = (x_1, x_2), y = (y_1, y_2).$$

This is the metric that corresponds to the "distance" we use in our everyday language and the distance that we use to solve problems involving Euclidan geometry.

**Lemma 1** (Cauchy-Schwarz Inequality) Let  $x_1, x_2$  and  $y_1, y_2$  be real numbers. Then

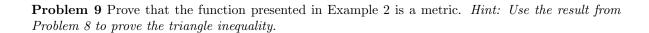
$$|x_1y_1 + x_2y_2| \le \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}.$$

**Problem 7** (Bonus) Prove Lemma 1.

Problem 8 Use the Cauchy-Schwarz inequality to prove that

$$(x_1 + y_1)^2 + (x_2 + y_2)^2 \le \left(\sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}\right)^2$$

where  $x_1, x_2, y_1$ , and  $y_2$  are any real numbers.



**Example 3** (The Taxicab Metric on  $\mathbb{R}^2$ )

$$d_{tc}(x,y) := |x_1 - y_1| + |x_2 - y_2|$$

for 
$$x = (x_1, x_2), y = (y_1, y_2).$$

**Problem 10** Prove that the function presented in Example 3 is a metric. For the triangle inequality condition, use the inequality from Problem 5.

**Problem 11** Another name for this metric is the Manhattan metric. Manhattan is a part of New York City where the streets form a grid; from the point of view of a bird, the streets of Manhattan look a lot like graph paper! Explain why either Taxicab or Manhattan are applicable names for this metric. Geometrically, how does this differ from the Euclidean metric?

We will now look at applications to urban geography using the Taxicab metric.

**Definition 3** Given a set of points A, B, C, ... the point or points P at which

$$d_{tc}(P,A) + d_{tc}(P,B) + d_{tc}(P,C) + \dots$$

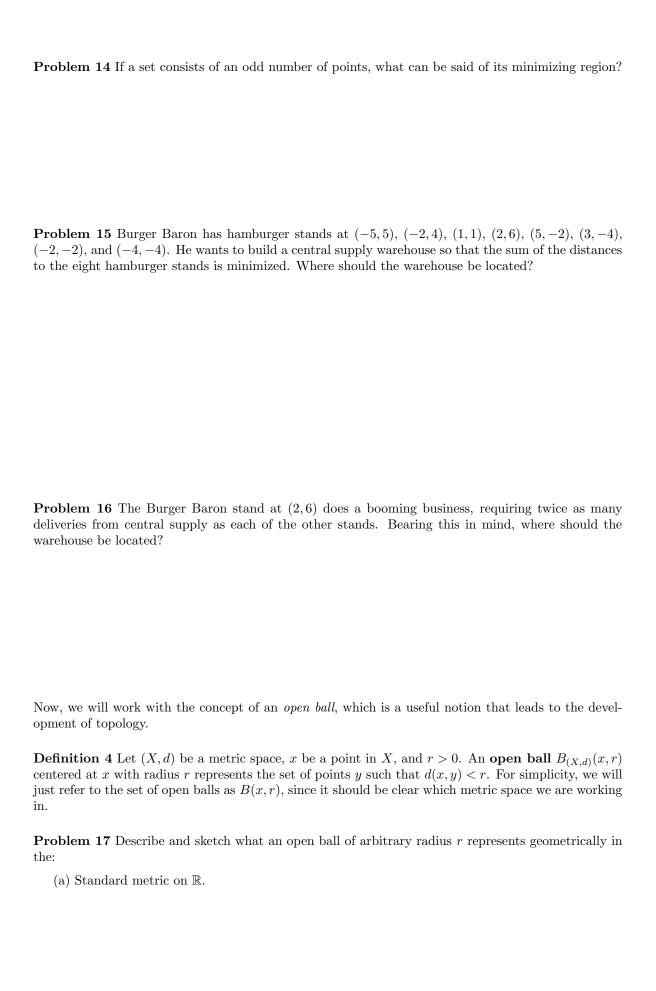
is as small as possible will be called the **minimizing region** of the set.

**Problem 12** Locate the minimizing region for the following sets:

(a) 
$$A = (-2, 3), B = (1, -4).$$

(b) 
$$A = (-3, 4), B = (4, 3), C = (0, -2).$$

**Problem 13** Alice, Bruno, and Clyde have to walk to A = (-3, -1), B = (3, 3), and C = (0, -3), respectively. Where should they live so that the sum of the distances they have to walk is a minimum?



(b) Euclidean Metric on  $\mathbb{R}^2$ .

(c) Taxicab Metric on  $\mathbb{R}^2$ .

**Problem 18** Determine which of the following functions are metrics. If the given function is not a metric, provide a counterexample.

(a) 
$$d_1(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \quad x, y \in \mathbb{R}$$

(b) 
$$d_2(x,y) = (x-y)^2 \quad x, y \in \mathbb{R}$$

(c) 
$$d_3(a,b) = \max_i |a_i - b_i| \quad a, b \in \mathbb{R}^n$$

(c) 
$$d_4(x,y) = |sin(x) - sin(y)| \quad x, y \in \mathbb{R}$$

