ORMC AMC 10/12 Group
Week 2: Functions

October 8, 2023

1 Warm-up Exercises

1. **(1993 AHSME #12)** If $f(2x) = \frac{2}{2+x}$ for all $x > 0$, what is $2f(x)$?

   We simply substitute $x/2$ into the expression for $f(2x)$, and then multiply the result by 2:

$$2f(x) = 2f(2 \cdot x/2) = 2 \cdot \frac{2}{2 + x/2} = 2 \cdot \frac{4}{4+x} = \frac{8}{4+x}.$$

2. Suppose $t(x) = ax^4 + bx^2 + x + 5$ where a and b are constants. If $t(-4) = 3$, find $t(4)$.

   Notice that x^4, x^2, x^0 do not change between 4 and -4. So, we only need to account for the difference in x, which goes from $-4 \to 4$, a change of $+8$. So, $t(4) = t(-4) + 8 = 3 + 8 = 11$.

3. For each of the following functions, find the x- and y- intercepts of its graph. Sketch the graphs of the functions.

   (a) $f(x) = (x-6)^2 - 4$

$$\text{x-intercepts: } (x-6)^2 - 4 = x^2 - 12x + 36 - 4 = (x-8)(x-4) = 0 \implies \{8, 4\}$$
$$\text{y-intercepts: } (0-6)^2 - 4 = 36 - 4 = 32$$

   (b) $f(x) = 4$

$$\text{x-intercepts: } 4 = 0 \implies \text{ no x-intercepts}$$
$$\text{y-intercepts: horizontal line at } y = 4 \implies \{4\}$$

   (c) $f(x) = \sqrt{x+5} - 3$

$$\text{x-intercepts:} \sqrt{x+5} - 3 = 0 \implies x + 5 = 9 \implies \{4\}$$
$$\text{y-intercepts:} \sqrt{5} - 3$$

   (d) $f(x) = |x+2| + 1$

$$\text{x-intercepts:} |x+2| + 1 = 0 \implies |x+2| = -1 \implies \text{ no x=intercepts}$$
$$\text{y-intercepts:} |2| + 1 = 3$$

4. Suppose $f(x) = 3x + 7$ and $g(x) = 2\sqrt{x-3}$

(a) Find $f(g(3))$ and $g(f(3))$.

$$f(g(3)) = 3(2\sqrt{3-3}) + 7 = 7$$
$$g(f(3)) = 2\sqrt{(3 \cdot 3 + 7) - 3} = 2\sqrt{9 + 7 - 3} = 2\sqrt{13}.$$

(b) Find a, if $f(g(a)) = 25$.

$$25 = 3(2\sqrt{a-3}) + 7 \implies 18 = 3(\sqrt{a-3}) \implies 6 = \sqrt{a-3} \implies 36 = a - 3 \implies a = 39.$$

(c) Is $g(f(-7))$ defined? Why or why not?

$$g(f(-7)) = 2\sqrt{(3(-7) + 7) - 3} = 2\sqrt{(-14) - 3},$$

but the square root of a negative number is not a real number, so it is not defined.

(d) Suppose that $h = g \circ f$. What is the domain of h? The domain of g is $x \geq 3$, so we are looking for the set where $f(x) \geq 3$, since $f(x)$ is being used as the input of g. So the domain is

$$3x + 7 \geq 3 \implies 3x \geq -4 \implies x \geq -\frac{4}{3}.$$

2 Function Basics

A function is an expression f that takes an input x, and outputs at most one value $y = f(x)$. Functions may take multiple inputs, like $f(x, y, z)$. In this case, there can be at most one output per ordered tuple (x, y, z).

1. **domain**: The input set X of values x that correspond to at least one output $f(x)$.

2. **codomain**: The set Y of values that $f(x)$ might possibly output on an arbitrary input x.

3. **range**: A subset of the codomain, consisting of all values $f(x)$ which are outputs of the function for some input x in the domain. When the domain is X, the range is written as $f(X)$.

Example: $f : \mathbb{Z} \mapsto \mathbb{Z}$ defined by $f(x) = |x|$ would have domain and codomain $\mathbb{Z}$, and range $\mathbb{N} \cup \{0\}$.

Also important is **composition**, applying one function to the output of another to create a new function, i.e. $h(x) = f(g(x))$. Often written as $f \circ g(x)$.

Functions are often defined explicitly, like $f(x) = \frac{1}{1-x-x^2}$. They may also be defined **recursively**, meaning that it is defined in terms of other outputs, and some base case(s). If you're familiar with inductive proofs, the idea is similar.

For example, we can define a function f on the integers as:

$$F(0) = 0,\ F(1) = 1,\ \text{ and } F(n+2) = F(n+1) + F(n) \text{ for } n \geq 0.$$

(Bonus challenge if you finish the worksheet early: find an explicit definition for $F(n)$)

2.1 Examples

1. Suppose $f(4) = 8$. Notice that the point $(2, 8)$ is on the graph of $f(2x)$. Notice also that the point $(5, 8)$ is on the graph of $f(2x - 6)$. The point $(5, 8)$ is 3 units to the right of $(2, 8)$. Is every point on $f(2x - 6)$ 3 units to the right of a point on $f(2x)$? Does this apply to any function? Why or why not?

   Yes, every point on $f(2x - 6)$ is 3 units to the right of some point on $f(2x)$. This may be slightly clearer when we write $f(2x - 6) = f(2(x - 3))$. The significance of this is that if (a, b) is on the graph of $f(2(x - 3))$, then we can easily see that $(a - 3, b)$ must be on the graph of $f(2x)$. This is true in general– when we isolate x and subtract some number t from it, that shifts the graph to the right by t units. Similarly, if we add t to x, that would shift the graph to the *left* by t units. As a follow up: what does the 2 do to the graph? what if we were to add/subtract the 3 *outside* the $f(x)$ instead of inside? What if we multiplied by the 2 outside the $f(x)$? It is important to understand what each of these does in terms of transforming the graph.

2. Let $f(x) = 3x-1$. Find $f(f(x))$ and $f(f(f(x)))$. (We often write these as $f^2(x)$ and $f^3(x)$, respectively, for convenience)

   $f(f(x)) = 3(3x-1)-1 = 9x-3-1 = 9x-4$, and $f(f(f(x))) = 3(9x-4)-1 = 27x-12-1 = 27x-13$.

3. Let $f(x) = \frac{-5-x}{3-2x}$. Find $f^{-1}(x)$

   We do this by substituting $f^{-1}(x)$ for x in the equation, and then solving for it:

$$x = \frac{-5 - f^{-1}x}{3 - 2f^{-1}(x)} \implies 3x - 2xf^{-1}(x) = -5 - f^{-1}(x) \implies 3x + 5 = (2x - 1)f^{-1}(x)$$

$$\implies f^{-1}(x) = \frac{3x + 5}{2x - 1}.$$

4. If $f(x) = ax^3 + bx^2 + cx + d$ and $f(-1) = 0, f(0) = 2, f(1) = 0$, then what is b?

   The easiest way to do this is to notice that if $f(-1) = 0$ and $f(1) = 0$, then that means $(x-1)(x+1) = x^2 - 1$ divides $f(x)$. In particular, we can write $f(x) = r(x^2 - 1)(tx + s)$. This means that b must be rs, and d must be $-rs$. But we know that d is the y-intercept, which is given to us as 2. So, it follows that $b = -d = -2$.

2.2 Exercises

1. Suppose $g(x) = ax + b$, where a, b are real numbers. Find all possible pairs (a, b) for which $g^2(x) = 9x + 28$.

   We have $g^2(x) = a(ax + b) + b = a^2x + ab + b$. So, we want $a^2 = 9, b(a + 1) = 28$. This means that $a = \pm 3$, which means that $b = 7$ or -14, respectively. So, the possible pairs are $(3, 7), (-3, -14)$.

2. If $f(a) = a - 2$ and $F(a, b) = b^2 + a$ compute $F(3, f(4))$.

   We have $F(3, f(4)) = f(4)^2 + 3 = (4 - 2)^2 + 3 = 4 + 3 = 7$.

3. If $f(x) = \frac{x}{x+1}$, what is $f^4(2009)$?

   Notice that $f(a/b) = \frac{a/b}{a/b+1} = \frac{a/b}{a/b+b/b} = \frac{a}{a+b}$. So, we will have $f^4(x) = \frac{x}{4x+1} \implies f^4(2009) = \frac{2009}{8037}$

4. **(2008 AMC 12A #12)** A function f has domain $[0, 2]$ and range $[0, 1]$. What are the domain and range, respectively, of the function g defined by $g(x) = 1 - f(x + 1)$?

   $g(x)$ is defined if $f(x + 1)$ is defined. Thus the domain is all $x | x + 1 \in [0, 2] \to x \in [-1, 1]$.

   Since $f(x + 1) \in [0, 1]$, $-f(x + 1) \in [-1, 0]$. Thus $g(x) = 1 - f(x + 1) \in [0, 1]$ is the range of $g(x)$.

   Thus the answer is $[-1, 1], [0, 1]$

5. **(1984 AHSME #16)** The function $f(x)$ satisfies $f(2 + x) = f(2 - x)$ for all real numbers x. If the equation $f(x) = 0$ has four distinct real roots, what is the sum of those roots?

   Let one of the roots be r_1. Also, define x such that $2 + x = r_1$. Thus, we have $f(2 + x) = f(r_1) = 0$ and $f(2 + x) = f(2 - x)$. Therefore, we have $f(2 - x) = 0$, and $2 - x$ is also a root. Let this root be r_2. The sum $r_1 + r_2 = 2 + x + 2 - x = 4$. Similarly, we can let r_3 be a root and define y such that $2 + y = r_3$, and we will find $2 - y$ is also a root, say, r_4, so $r_3 + r_4 = 2 + y + 2 - y = 4$. Therefore, $r_1 + r_2 + r_3 + r_4 = 4 + 4 = 8$

6. **(1983 AHSME #18)** Let f be a polynomial function such that for all real x, $f(x^2 + 1) = x^4 + 5x^2 + 3$. For all real x, what is $f(x^2 - 1)$?

   Let $y = x^2 + 1$. Then $x^2 = y - 1$, so we can write the given equation as

$$\begin{aligned} f(y) &= x^4 + 5x^2 + 3 \\ &= (x^2)^2 + 5x^2 + 3 \\ &= (y-1)^2 + 5(y-1) + 3 \\ &= y^2 - 2y + 1 + 5y - 5 + 3 \\ &= y^2 + 3y - 1. \end{aligned}$$

Then substituting $x^2 - 1$ for y, we get

$$\begin{aligned} f(x^2-1) &= (x^2-1)^2 + 3(x^2-1) - 1 \\ &= x^4 - 2x^2 + 1 + 3x^2 - 3 - 1 \\ &= x^4 + x^2 - 3. \end{aligned}$$

7. Let $g(x) = \frac{ax+b}{cx+d}$, where a, b, c, d are positive real numbers such that $ad \neq bc$. Which of $\frac{a}{b}$, $\frac{a}{c}$, and $\frac{a}{d}$ cannot be in the domain of g^{-1}?

8. **(2000 AMC 12 #15)** Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

   Let $y = \frac{x}{3}$; then $f(y) = (3y)^2 + 3y + 1 = 9y^2 + 3y + 1$. Thus $f(3z) - 7 = 81z^2 + 9z - 6 = 3(9z-2)(3z+1) = 0$, and $z = -\frac{1}{3}, \frac{2}{9}$. These sum up to $\boxed{-\frac{1}{9}}$

9. **(2004 AMC 12B #13)** If $f(x) = ax + b$ and $f^{-1}(x) = bx + a$ with a and b real, what is the value of $a + b$?

   Since $f(f^{-1}(x)) = x$, it follows that $a(bx + a) + b = x$, which implies $abx + a^2 + b = x$. This equation holds for all values of x only if $ab = 1$ and $a^2 + b = 0$.

   Then $b = -a^2$. Substituting into the equation $ab = 1$, we get $-a^3 = 1$. Then $a = -1$, so $b = -1$, and

$$f(x) = -x - 1.$$

   Likewise

$$f^{-1}(x) = -x - 1.$$

   These are inverses to one another since

$$f(f^{-1}(x)) = -(-x-1) - 1 = x + 1 - 1 = x.$$

$$f^{-1}(f(x)) = -(-x-1) - 1 = x + 1 - 1 = x.$$

   Therefore $a + b = \boxed{-2}$.

10. **(2017 AMC 12A #7)** Define a function on the positive integers recursively by $f(1) = 2$, $f(n) = f(n-1) + 1$ if n is even, and $f(n) = f(n-2) + 2$ if n is odd and greater than 1. What is $f(2017)$?

    This is a recursive function, which means the function refers back to itself to calculate subsequent terms. To solve this, we must identify the base case, $f(1) = 2$. We also know that when n is odd, $f(n) = f(n-2) + 2$. Thus we know that $f(2017) = f(2015) + 2$. Thus we know that n will always be odd in the recursion of $f(2017)$, and we add 2 each recursive cycle, which there are 1008 of. Thus the answer is $1008 * 2 + 2 = 2018$, which is answer $\boxed{\textbf{(B)}}$. Note that when you write out a few numbers, you find that $f(n) = n + 1$ for any n, so $f(2017) = 2018$

11. **(2015 AMC 12A #18)** The zeros of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

    Let m and n be the roots of $x^2 - ax + 2a$

Notice that if we write out the factorization with m and n, it would look like $x^2-ax+2a=(x-m)(x-n)$, so $n+m=a$ and $mn=2a$

Substituting gets us $n+m=\frac{mn}{2}$

$2n-mn+2m=0$

Using Simon's Favorite Factoring Trick:

$n(2-m)+2m=0$

$-n(2-m)-2m=0$

$-n(2-m)-2m+4=4$

$(2-n)(2-m)=4$

This means that the values for (m,n) are $(0,0),(4,4),(3,6),(1,-2)$ giving us a values of $-1,0,8$, and 9. Adding these up gets $\boxed{16}$.

12. **(2014 AMC 12A #18)** The domain of the function $f(x)=\log_{\frac{1}{2}}(\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}} x))))$ is an interval of length $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

   For simplicity, let $a=\log_{\frac{1}{16}} x, b=\log_{16} a, c=\log_{\frac{1}{4}} b$, and $d=\log_4 c$.

   The domain of $\log_{\frac{1}{2}} x$ is $x\in(0,\infty)$, so $d\in(0,\infty)$. Thus, $\log_4 c\in(0,\infty)\Rightarrow c\in(1,\infty)$. Since $c=\log_{\frac{1}{4}} b$ we have $b\in\left(0,\left(\frac{1}{4}\right)^1\right)=\left(0,\frac{1}{4}\right)$. Since $b=\log_{16} a$, we have $a\in(16^0,16^{1/4})=(1,2)$. Finally, since $a=\log_{\frac{1}{16}} x$, $x\in\left(\left(\frac{1}{16}\right)^2,\left(\frac{1}{16}\right)^1\right)=\left(\frac{1}{256},\frac{1}{16}\right)$.

   The length of the x interval is $\frac{1}{16}-\frac{1}{256}=\frac{15}{256}$ and the answer is $\boxed{271}$

13. **(2015 AMC 12B #18)** For every composite positive integer n, define $r(n)$ to be the sum of the factors in the prime factorization of n. For example, $r(50)=12$ because the prime factorization of 50 is 2×5^2, and $2+5+5=12$. What is the range of the function r, $\{r(n) : n \text{ is a composite positive integer}\}$?

14. **(2015 AMC 12B #20)** For every positive integer n, let $\text{mod}_5(n)$ be the remainder obtained when n is divided by 5. Define a function $f : \{0,1,2,3,\dots\} \times \{0,1,2,3,4\} \to \{0,1,2,3,4\}$ recursively as follows:

$$f(i,j) = \begin{cases} \text{mod}_5(j+1) & \text{if } i = 0 \text{ and } 0 \le j \le 4, \\ f(i-1,1) & \text{if } i \ge 1 \text{ and } j = 0, \text{ and} \\ f(i-1, f(i,j-1)) & \text{if } i \ge 1 \text{ and } 1 \le j \le 4. \end{cases}$$

What is $f(2015,2)$?

15. **(2021 AMC 12A #18)**Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

(A) $\frac{17}{32}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{7}{9}$ **(D)** $\frac{7}{6}$ **(E)** $\frac{25}{11}$

Think backwards. The range is the same as the numbers y that can be expressed as the sum of two or more prime positive integers.

The lowest number we can get is $y = 2 + 2 = 4$. For any number greater than 4, we can get to it by adding some amount of 2's and then possibly a 3 if that number is odd. For example, 23 can be obtained by adding 2 ten times and adding a 3; this corresponds to the argument $n = 2^{10} \times 3$. Thus our answer is $\boxed{\text{the set of integers greater than 3}}$.

16. **(2018 AMC 10B #20)** A function f is defined recursively by $f(1) = f(2) = 1$, and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \ge 3$. Find $f(2018)$.

For all integers $n \ge 7$, note that

$$\begin{aligned} f(n) &= f(n-1) - f(n-2) + n \\ &= [f(n-2) - f(n-3) + n - 1] - f(n-2) + n \\ &= -f(n-3) + 2n - 1 \\ &= -[f(n-4) - f(n-5) + n - 3] + 2n - 1 \\ &= -f(n-4) + f(n-5) + n + 2 \\ &= -[f(n-5) - f(n-6) + n - 4] + f(n-5) + n + 2 \\ &= f(n-6) + 6. \end{aligned}$$

It follows that

$$\begin{aligned} f(2018) &= f(2012) + 6 \\ &= f(2006) + 12 \\ &= f(2000) + 18 \\ &\vdots \\ &= f(2) + 2016 \\ &= \boxed{\textbf{(B) } 2017}. \end{aligned}$$

17. **(2021 AMC 12A #25)**Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1, d(2) = 2$, and $d(12) = 6$. (This function is known as the divisor function.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of N?

The question statement asks for the value of N that maximizes $f(N)$. Let N start out at 1; we will find what factors to multiply N by, in order for N to maximize the function.

First, we will find what power of 2 to multiply N by. If we multiply N by 2^a, the numerator of f, $d(N)$, will multiply by a factor of $a+1$; this is because the number 2^a has $a+1$ divisors. The denominator, $\sqrt[3]{N}$, will simply multiply by $\sqrt[3]{2^a}$. Therefore, the entire function multiplies by a factor of $\frac{a+1}{\sqrt[3]{2^a}}$. We want to find the integer value of a that maximizes this value. By inspection, this is 3. Therefore, we multiply N by 8; right now, N is 8.

Next, we will find what power of 3 to multiply N by. Similar to the previous step, we wish to find the integer value of a that maximizes $\frac{a+1}{\sqrt[3]{3^a}}$. This value, also by inspection, is 2.

We can repeat this step on the rest of the primes to get

$$N = 2^3 \cdot 3^2 \cdot 5 \cdot 7$$

but from 11 on, $a = 0$ will maximize the value of the function, so the prime is not a factor in N. We evaluate N to be 2520, so the answer is $\boxed{9}$.