ORMC AMC 10/12 Group
Week 2: Functions

October 8, 2023

1 Warm-up Exercises

1. **(1993 AHSME #12)** If $f(2x) = \frac{2}{2+x}$ for all $x > 0$, what is $2f(x)$?

2. Suppose $t(x) = ax^4 + bx^2 + x + 5$ where a and b are constants. If $t(-4) = 3$, find $t(4)$.

3. For each of the following functions, find the x- and y- intercepts of its graph. Sketch the graphs of the functions.

   (a) $f(x) = (x-6)^2 - 4$ (b) $f(x) = 4$

   (c) $f(x) = \sqrt{x+5} - 3$ (d) $f(x) = |x+2| + 1$

4. Suppose $f(x) = 3x + 7$ and $g(x) = 2\sqrt{x-3}$

   (a) Find $f(g(3))$ and $g(f(3))$.
   (b) Find a, if $f(g(a)) = 25$.
   (c) Is $g(f(-7))$ defined? Why or why not?
   (d) Suppose that $h = g \circ f$. What is the domain of h?

2 Function Basics

A function is an expression f that takes an input x, and outputs at most one value $y = f(x)$. Functions may take multiple inputs, like $f(x, y, z)$. In this case, there can be at most one output per ordered tuple (x, y, z).

1. **domain**: The input set X of values x that correspond to at least one output $f(x)$.
2. **codomain**: The set Y of values that $f(x)$ might possibly output on an arbitrary input x.
3. **range**: A subset of the codomain, consisting of all values $f(x)$ which are outputs of the function for some input x in the domain. When the domain is X, the range is written as $f(X)$.

Example: $f : \mathbb{Z} \mapsto \mathbb{Z}$ defined by $f(x) = |x|$ would have domain and codomain $\mathbb{Z}$, and range $\mathbb{N} \cup \{0\}$.

Also important is **composition**, applying one function to the output of another to create a new function, i.e. $h(x) = f(g(x))$. Often written as $f \circ g(x)$.

Functions are often defined explicitly, like $f(x) = \frac{1}{1-x-x^2}$. They may also be defined **recursively**, meaning that it is defined in terms of other outputs, and some base case(s). If you're familiar with inductive proofs, the idea is similar.

For example, we can define a function f on the integers as:

$$F(0) = 0,\ F(1) = 1,\ \text{ and } F(n+2) = F(n+1) + F(n) \text{ for } n \geq 0.$$

(Bonus challenge if you finish the worksheet early: find an explicit definition for $F(n)$)

2.1 Examples

1. Suppose $f(4) = 8$. Notice that the point $(2, 8)$ is on the graph of $f(2x)$. Notice also that the point $(5, 8)$ is on the graph of $f(2x - 6)$. The point $(5, 8)$ is 3 units to the right of $(2, 8)$. Is every point on $f(2x - 6)$ 3 units to the right of a point on $f(2x)$? Does this apply to any function? Why or why not?
2. Let $f(x) = 3x - 1$. Find $f(f(x))$ and $f(f(f(x)))$. (We often write these as $f^2(x)$ and $f^3(x)$, respectively, for convenience)
3. Let $f(x) = \frac{-5-x}{3-2x}$. Find $f^{-1}(x)$
4. If $f(x) = ax^3 + bx^2 + cx + d$ and $f(-1) = 0, f(0) = 2, f(1) = 0$, then what is b?

2.2 Exercises

1. Suppose $g(x) = ax + b$, where a, b are real numbers. Find all possible pairs (a, b) for which $g^2(x) = 9x + 28$.

2. If $f(a) = a - 2$ and $F(a, b) = b^2 + a$ compute $F(3, f(4))$.

3. If $f(x) = \frac{x}{x+1}$, what is $f^4(2009)$?

4. **(2008 AMC 12A #12)** A function f has domain $[0, 2]$ and range $[0, 1]$. What are the domain and range, respectively, of the function g defined by $g(x) = 1 - f(x + 1)$?

5. **(1984 AHSME #16)** The function $f(x)$ satisfies $f(2 + x) = f(2 - x)$ for all real numbers x. If the equation $f(x) = 0$ has four distinct real roots, what is the sum of those roots?

6. **(1983 AHSME #18)** Let f be a polynomial function such that for all real x, $f(x^2+1) = x^4+5x^2+3$. For all real x, what is $f(x^2 - 1)$?

7. Let $g(x) = \frac{ax+b}{cx+d}$, where a, b, c, d are positive real numbers such that $ad \neq bc$. Which of $\frac{a}{b}$, $\frac{a}{c}$, and $\frac{a}{d}$ cannot be in the domain of g^{-1}?

8. **(2000 AMC 12 #15)** Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

9. **(2004 AMC 12B #13)** If $f(x) = ax + b$ and $f^{-1}(x) = bx + a$ with a and b real, what is the value of $a + b$?

10. **(2017 AMC 12A #7)** Define a function on the positive integers recursively by $f(1) = 2$, $f(n) = f(n-1) + 1$ if n is even, and $f(n) = f(n-2) + 2$ if n is odd and greater than 1. What is $f(2017)$?

11. **(2015 AMC 12A #18)** The zeros of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

12. **(2014 AMC 12A #18)** The domain of the function $f(x) = \log_{\frac{1}{2}}(\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}} x))))$ is an interval of length $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

13. **(2015 AMC 12B #18)**For every composite positive integer n, define $r(n)$ to be the sum of the factors in the prime factorization of n. For example, $r(50) = 12$ because the prime factorization of 50 is 2×5^2, and $2 + 5 + 5 = 12$. What is the range of the function r, $\{r(n) : n \text{ is a composite positive integer}\}$?

14. **(2015 AMC 12B #20)** For every positive integer n, let $\text{mod}_5(n)$ be the remainder obtained when n is divided by 5. Define a function $f : \{0, 1, 2, 3, \dots\} \times \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$ recursively as follows:

$$f(i,j) = \begin{cases} \text{mod}_5(j+1) & \text{if } i = 0 \text{ and } 0 \le j \le 4, \\ f(i-1, 1) & \text{if } i \ge 1 \text{ and } j = 0, \text{ and} \\ f(i-1, f(i, j-1)) & \text{if } i \ge 1 \text{ and } 1 \le j \le 4. \end{cases}$$

What is $f(2015, 2)$?

15. **(2021 AMC 12A #18)**Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

(A) $\frac{17}{32}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{7}{9}$ **(D)** $\frac{7}{6}$ **(E)** $\frac{25}{11}$

16. **(2018 AMC 10B #20)** A function f is defined recursively by $f(1) = f(2) = 1$, and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \ge 3$. Find $f(2018)$.

17. **(2021 AMC 12A #25)**Let $d(n)$ denote the number of positive integers that divide n, including 1 and n. For example, $d(1) = 1, d(2) = 2$, and $d(12) = 6$. (This function is known as the divisor function.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer N such that $f(N) > f(n)$ for all positive integers $n \ne N$. What is the sum of the digits of N?