

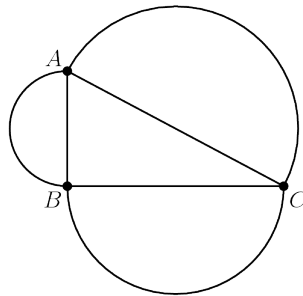
Algebra!

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1 Warm Up

Angle ABC of $\triangle ABC$ is a right angle. The sides of $\triangle ABC$ are the diameters of semicircles as shown. The area of the semicircle on \overline{AB} equals 8π , and the arc of the semicircle on \overline{AC} has length 8.5π . What is the radius of the semicircle on \overline{BC} ?



- (A) 7 (B) 7.5 (C) 8 (D) 8.5 (E) 9

Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?

- (A) 3 (B) 6 (C) 9 (D) 12 (E) 18

2 Overview

The AMC 8 has a variety of topics, but the simplest and most frequent type of questions are ones that involve algebra. Since a lot of these topics tend to be simpler, we will be covering a few today! In the coming weeks, we will also study combinatorics, number theory, and geometry. All of these concepts will use what we learn in algebra, so pay attention and make sure you really understand!

3 Ratios and Rates

Ratios compare groups to each other quantitatively. For example, if you do a survey of people who like cats or dogs, you can set up a ratio comparing the number of people who like cats, c , to the number of people who like dogs, d . When we want to write the ratio, we can write it as

$$d : c$$

The little colon symbol denotes the split of the two groups. Ratios can be between two or more groups, so we can get ratios for things like angle measures of shapes or students in each grade of a school. Try the problem below!

There are 270 students at Colfax Middle School, where the ratio of boys to girls is 5 : 4. There are 180 students at Winthrop Middle School, where the ratio of boys to girls is 4 : 5. The two schools hold a dance and all students from both schools attend. What fraction of the students at the dance are girls?

A rate is simply a quantity measured against another quantity. You use rates every day to describe many activities. Usually, you compare the amount of time it takes to complete x amount of things. Some common rates include:

- miles per hour
- cost per ounce
- calories per serving
- points per question

Since they are so common in real life, you will see them on the AMC 8! You will most probably see work rates (like finishing some activity in a certain amount of time).

Work rates can sometimes be trickier. Luckily, we have a very easy formula for work rates.

$$\text{Work}=\text{Rate}\times\text{Time}$$

Equivalently,

$$\begin{aligned}\text{Rate}&=\frac{\text{Work}}{\text{Time}} \\ \text{Time}&=\frac{\text{Work}}{\text{Rate}}\end{aligned}$$

In certain problems, we can "stack" rates. This happens in questions like the following:

10 workers from a company each working at a constant rate can build 10 houses in 12 years. However, after 6 years, 5 of them retire. The other workers continue working at the same original rate. The project is falling behind, so 2 years later, the company hires 10 more workers who each work twice as fast as the original workers. After how many total years will the 10 houses be complete?

Notice that in order to solve the question, we have to find the one worker's rate of building houses so that we can continue despite the setback this construction project faces with retirees and new hires.

4 Speed, Distance, Time

Speed, distance, and time can all be related in one simple equation! Speed is a rate (given in miles per hour, most times). We can use the below equation and it's manipulations.

$$\begin{aligned}\text{Distance}&=\text{Speed}\times\text{Time} \\ \text{Speed}&=\frac{\text{Distance}}{\text{Time}} \\ \text{Time}&=\frac{\text{Distance}}{\text{Speed}}\end{aligned}$$

When approaching problems involving speed, it helps to ask a couple questions. First, ask what information is given. Then, ask how you can calculate the number of minutes. Finally, think about how many minutes the certain activity will actually take. Use this strategy and the equations above to solve the following practice problem!

Kylar needs to get to his office 40 miles away. If he drives at 60 miles per hour, how many minutes will it take him to get to work?

We can also use a similar formula to find the average speed for a total distance covered over a total amount of time.

$$\text{Average Speed} = \frac{\text{TotalDistance}}{\text{TotalTime}}$$

To use this formula, you may have to add multiple distance to find the total distance, and use a speed to find the total time. Try the following practice problem!

Cole is running along a 2 mile path at 8 miles per hour and back on the same path. On the way back he is tired so for after every 1 mile he runs on the way back, his speed is instantly reduced by 1 mile per hour. What is his average speed throughout the whole trip?

5 Mean, Median, Mode

Mean, median and mode may seem like a simple topic, but let's review it and then try a practice problem!

Mean is the average of a set of numbers. To find the mean, you add all the numbers and divide this sum by the number of values.

Median is the middle number of the set. You have to arrange the numbers in order from least to greatest and select the number in the very middle. If there is an even amount of numbers, choose the two middle numbers and average them.

Mode is the value that occurs the most in the sequence. In some cases, you may have no modes. You also may have multiple modes!

The range is just the smallest number subtracted from the largest number in the set. It gives you the "distance" from the smallest number to the largest number.

Let's try a practice problem!

What is the median of the following list of 4040 numbers?

$$1, 2, 3, \dots, 2020, 1^2, 2^2, 3^2, \dots, 2020^2$$

Even though the mean, median, mode and range are relatively simple topics, they can be made into more difficult problems. You'll see more of these in the practice section below. For now, let's try another practice problem!

The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. The largest integer that can be an element of this collection is what?

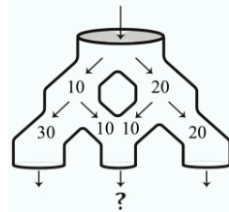
6 Practice!

A number of students from Fibonacci Middle School are taking part in a community service project. The ratio of 8th-graders to 6th-graders is $5 : 3$, and the ratio of 8th-graders to 7th-graders is $8 : 5$. What is the smallest number of students that could be participating in the project?

All of Aurora's marbles are blue, red, green, or yellow. $\frac{1}{3}$ of her marbles are blue, $\frac{1}{4}$ of them are red, and 6 of them are green. What is the smallest number of yellow marbles Aurora can have?

Jakub can complete $\frac{5}{6}$ of a job in $\frac{3}{4}$ of the time that it takes William to do the whole job. What is the ratio of the rate at which Jakub works to the rate at which William works? Express your answer as a common fraction.

Ten thousand marbles are released into the top pipe as shown and roll down the pipe system. Anytime the pipe forks, the marbles split in proportion to the cross-sectional areas of the pipes. All pipes have circular cross-sections with diameters as indicated in the figure. How many marbles exit through the bottom, middle pipe?



Two jars each contain the same number of marbles, and every marble is either blue or green. In Jar 1 the ratio of blue to green marbles is $9 : 1$, and the ratio of blue to green marbles in Jar 2 is $8 : 1$. There are 95 green marbles in all. How many more blue marbles are in Jar 1 than in Jar 2?

When the World Wide Web first became popular in the 1990s, download speeds reached a maximum of about 56 kilobits per second. Approximately how many minutes would the download of a 4.2-megabyte song have taken at that speed? (Note that there are 8000 kilobits in a megabyte.)

Rain takes five tests, each worth a maximum of 100 points. Her scores on the first three tests are 76, 94, and 87. In order to average 81 for all five tests, what is the lowest score she could earn on one of the other two tests?

Adhvik, Lauren, and Olivia got some candy eggs at a party. Adhvik had three times as many eggs as Lauren, and Lauren had twice as many eggs as Olivia. Adhvik decides to give some of his eggs to Lauren and Olivia so that all three will have the same number of eggs. What fraction of his eggs should Adhvik give to Lauren?

A car passes point A driving at a constant rate of 60 km per hour. A second car, traveling at a constant rate of 75 km per hour, passes the same point A a while later and then follows the first car. It catches the first car after traveling a distance of 75 km past point A. How many minutes after the first car passed point A did the second car pass point A?

Each day for four days, Mika traveled for one hour at a speed that resulted in her traveling one mile in an integer number of minutes. Each day after the first, her speed decreased so that the number of minutes to travel one mile increased by 5 minutes over the preceding day. Each of the four days, her distance traveled was also an integer number of miles. What was the total number of miles for the four trips?

Theo and Dylan start hiking from a trailhead toward a fire tower. Dylan is wearing a heavy backpack and walks slower. Theo starts walking at 4 miles per hour. Halfway to the tower, the trail becomes really steep, and Dylan slows down to 2 miles per hour. After reaching the tower, he immediately turns around and descends the steep part of the trail at 3 miles per hour. He meets Dylan at the halfway point. What was Dylan's average speed, in miles per hour, until they meet?

Ian drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S , then the average value (arithmetic mean) of the integers remaining is 32. If the least integer in S is also removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S . What is the average value of all the integers in the set S ?

Mr. Anish asked the members of his health class how many days last week they exercised for at least 30 minutes. The results are summarized in the following bar graph, where the heights of the bars represent the number of students. What was the mean number of days of exercise last week, rounded to the nearest hundredth, reported by the students in Mr. Anish's class?

