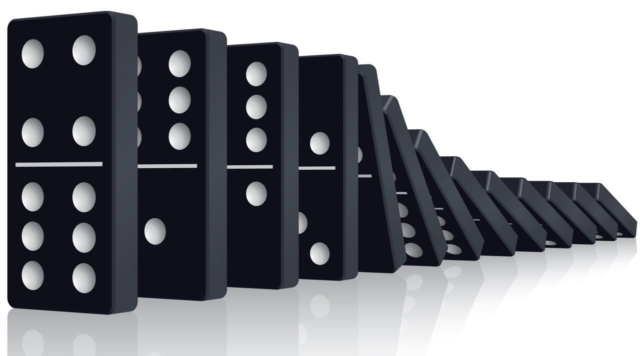
Induction and Combinatorics

Introduction

Suppose that we have an infinite list of related mathematical statements S_n where n are either natural numbers $1, 2, 3 \ldots$ or non-negative integers $0, 1, 2, 3 \ldots$ The first statement is called the *base case*. Suppose that S_1 is true. If we establish the *inductive step* by proving that S_n implies S_{n+1}, then we prove the validity of the statements S_n for any and all n. Indeed, $S_1 \Rightarrow S_2$, $S_2 \Rightarrow S_3$, $S_3 \Rightarrow S_4$, and so forth.

An example of mathematical induction is the *domino effect*. Imagine that we have an infinite set of dominoes lined up at equal distances along a straight line. Imagine further that the distance between the dominoes is short enough for a falling domino to force the fall of the next one.



Let us prove an infinite list of related statements

$$S_n = the\ nth\ domino\ falls$$

by induction.

The base case: the first domino falls. We prove it by inspection. Give the first domino a nudge and see what happens. If it falls, this proves the base case. If it doesn't, then the domino effect may not occur.

The inductive hypothesis: assume that S_n is true, the nth domino falls.

The inductive step: thinking S_n is true, prove that S_{n+1} is true as well. Proof – the falling nth domino forces the fall of the $n+1$ one.

This way, the fall of the first domino forces the fall of the second, the fall of the second forces the fall of the third, and so forth.

The following famous formula

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \tag{1}$$

was anecdotally discovered by Gauss at the age of three. (Outside of mathematical texts, the Greek letter Σ is pronounced as *sigma*. In mathematical texts, it means and reads a *sum*.) Here is Gauss's proof. Let us write down the sum twice, reversing the order of the summands the second time.

$$\begin{array}{lccccccccccc} \Sigma = & 1 & + & 2 & + & \ldots & + & (n-2) & + & (n-1) & + & n \\ \Sigma = & n & + & (n-1) & + & \ldots & + & 3 & + & 2 & + & 1 \end{array}$$

Adding the sums term-by-term produces the following.

$$2\Sigma = (n+1) + (n+1) + \ldots + (n+1) + (n+1) = n(n+1)$$

Dividing both sides by two proves (1).

To practice mathematical induction, let us use it to give a different proof to formula (1).

The base case: $n = 1$. The equality $1 = 1(1+1)/2$ is checked by inspection.

The inductive hypothesis: assume that formula (1) is true.

The inductive step: based on the assumption, prove that $1 + 2 + 3 + \ldots + n + (n+1) = (n+1)(n+2)/2$.

$$1 + 2 + \ldots + n + (n+1) = \underline{1 + 2 + \ldots + n} + (n+1) =$$

$$\frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2} + 1\right) = \frac{(n+1)(n+2)}{2}$$

The following formula

$$a + (a+q) + \ldots + (a + (n-1)q) = na + q\frac{n(n-1)}{2} \tag{2}$$

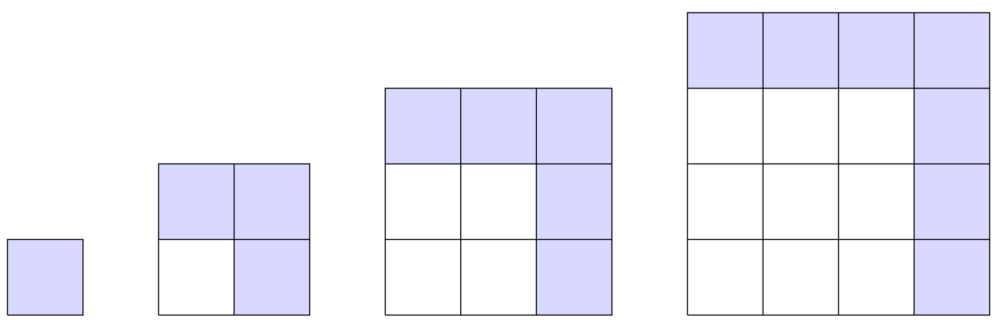
is a minor generalization of (1).

Problem 1 *Use mathematical induction to prove (2).*

Problem 2 *Use formula (2) for the sum of an arithmetic sequence to prove the following identity.*

$$1 + 3 + 5 + \ldots + (2n-1) = n^2 \tag{3}$$

Problem 3 *Consider the pictures below to prove formula (3) using geometry rather than algebra.*



Problem 4 *Use mathematical induction to prove the following.*

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{4}$$

Problem 5 *Experiment with sums of cubes of the form* $1^3+2^3+3^3+\ldots+n^3$ *for various natural* n*. Find a formula similar to (1) and (4). Then use mathematical induction to prove it.*

Binomial coefficients

Before moving on to the next problem, we will take a short detour to discuss binomial expansions. Our goal is to understand and prove the binomial formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{5}$$

The above equation looks scary because it has a lot of confusing notation so let's unpack it first. Recall the summation symbol $\sum$ which indicates that we are adding together a bunch of things. The right-hand side of formula (5) expands as

$$\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

But what does $\binom{n}{k}$ (read as "n choose k") mean? We introduce some definitions:

1. **Permutations:** Let n and k be non-negative integers such that $0 \leq k \leq n$. Then $P(n,k)$ is the number of ways to choose k objects out of n distinct objects in such a way that the order matters, i.e. the chosen objects form a list: the 1st chosen object, the 2nd chosen object, etc. It turns out that
$$P(n,k) = \frac{n!}{(n-k)!} \tag{6}$$
This is because we can calculate the number of ways to order k elements as follows: There are n ways to choose the first item, $n-1$ ways to choose the second item and so on. Therefore, there are $n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$ ways to order k elements.

2. **Combinations:** $C(n,k) = \binom{n}{k}$ = the number of ways to choose k objects out of n distinct objects in such a way that the order does

not matter, i.e. the chosen objects are considered as elements of a set rather than a list. We have a similar formula in this case.

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{7}$$

This is because we can find the number of ways to choose k items by first counting the number of ways to order them and then, since each set of k elements is included $k!$ times in this ordering (why?), we divide by $k!$ to compensate for this overcounting which gives us formula (7).

With these definitions, formula (5) tells us that $(a+b)^2 = a^2 + 2ab + b^2$ as expected.

Problem 6 *Compute $(a+b)^3$ and $(a+2b)^4$ using the binomial formula. Verify your answers by expanding the brackets.*

Now that we know how to interpret and use the formula, let's sketch its proof: When we expand $(a+b)^n$, each term is of the form $a^{n-k}b^k$ where k is between 0 and n. In other words, each term has degree n. We are interested in finding the coefficient of $a^{n-k}b^k$ for each k. Thinking about this a bit more, we can see that this reduces to a combinatorics problem.

Problem 7 *Given n plates, each with one apple and one berry, how many ways are there to take home n fruits such that we take k berries and $n-k$ apples?*

The answer is choose the k plates to take berries from ($\binom{n}{k}$ ways) and take apples from the rest of the plates. How does this relate to $(a+b)^n$? Each term in the expanded form of $(a+b)^n = (a+b)\cdots(a+b)$ comes from multiplying out and choosing different terms from the brackets. For example, we get the last term as b^n by choosing b all n times. If we choose b three times and a all other times we get the $a^{n-3}b^3$ term. How many ways are there to do this? There are $\binom{n}{3}$ ways of doing it! That's why we see $\binom{n}{3}a^{n-3}b^3$ appear in the expansion of $(a+b)^n$. In general, we get the coefficient of $a^{n-k}b^k$ by calculating the number of ways we can choose b in the expansion k times and a the other $n-k$ times. This is exactly $\binom{n}{k}$. By summing up all the terms, we get the binomial formula.

With the binomial formula and induction, we can prove that the following remarkable formula holds for the sum of the fourth powers of the first n natural numbers.

$$\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \tag{8}$$

Problem 8 *Prove (8).*

Problem 9 *Prove that* $n^n \geq (n+1)^{n-1}$.

Permutations and Combinations

In understanding the binomial expansion, we came across two important ideas: permutations and combinations. Before proceeding with more induction, let's try to solve a few more questions to get the hand of permutations and combinations.

Problem 10 *A group of 5 friends want to take a picture. In how many different ways can they stand in a line for the picture?*

Problem 11 *In a deck of standard 52 playing cards, how many different ways are there to select a 5-card hand?*

Problem 12 *In how many ways can you draw a hand of 5 cards that has a "full house"? (A full house consists of three cards of one rank and two cards of another rank. For instance, three 8s and two Jacks form a full house.)*

Problem 13 *In a class of 10 students, 3 will be selected to represent the class in a competition. In how many different ways can the 3 students be selected?*

Problem 14 *There are 8 books of different titles on a shelf. In how many ways can you select and arrange 3 of them side by side?*

Problem 15 *A committee of 5 members is to be formed from a group of 9 men and 6 women. In how many ways can the committee be formed if it should contain at least 3 women?*

Problem 16 *There are 10 points in a plane of which no three are collinear. A straight line is drawn by joining any two of these points. Find the number of triangles that can be formed using these lines.*

Problem 17 *In the word "PERMUTATIONS", how many different ways are there to arrange the letters such that all the vowels appear before all the consonants?*

Problem 18 *An office has 5 managers and 10 interns. They need to sit around a circular table with chairs numbered 1 through 15. In how many ways can they be seated if no two managers sit next to each other?*

Problem 19 *Derive the binomial formula by induction on n.*

Problem 20 *Give 3 different proofs that*

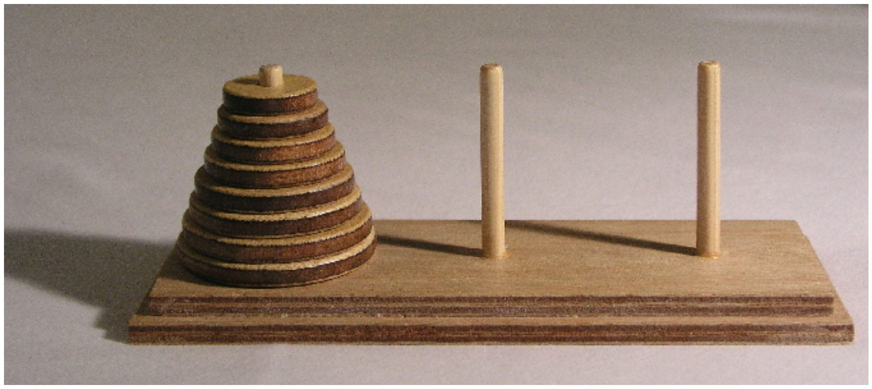
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Hint: a) can you use induction? b) what about the binomial formula? c) how many subsets does $\{1, \ldots, n\}$ have?

Now we return to induction by introducing the famous tower of Hanoi.

The tower of Hanoi

Hidden in the jungle near Hanoi, the capital city of Vietnam, there exists a Buddhist monastery where monks keep constantly moving golden disks from one diamond rod to another. There are 64 disks, all of different sizes, and three rods. Only one disk can be moved at a time and no larger disk can be placed on the top of a smaller one. Originally, all the disks were on one rod, say, the left one. At the end, they all must be moved to the right rod. When all the disks are moved, the world will come to an end.



The Hanoi tower with eight disks.

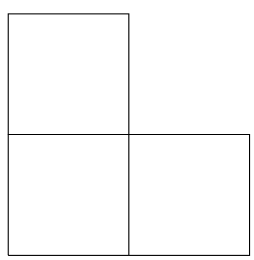
The tale was created by a French mathematician, Édouard Lucas, to promote the puzzle he had invented.

Problem 21 *Find the optimal, i.e. shortest, algorithm to solve the Hanoi tower puzzle with any number of disks.*

Problem 22 *The monks take shifts to move one disk per second non-stop, day and night. How much time would it take them to solve the puzzle with n disks?*

Problem 23 *Is the world going to come to an end any time soon? Why or why not?*

A *tromino* is the flat figure made of three squares drawn below.



Problem 24 *Split a tromino into four parts having the same shape and size.*

Problem 25 *Assume that the tromino squares are* 1×1*. Show that a* $2^n \times 2^n$ *square with a corner* 1×1 *square removed can be tiled by trominos.*

Problem 26 *Show that* $4^n + 1$ *is not a multiple of three* $\forall n \in \mathbb{N}$*.*

Problem 27 *Show that* $n^3 + 2n$ *is a multiple of three* $\forall n \in \mathbb{N}$*.*

Problem 28 *There are n coins on the table. Two players take turns taking either one or two coins off the table. The player to take the last coin wins. You are playing against an instructor, so you can choose whether to move first (P1) or second (P2). Find a way to always win.*