

---

# Random Walks and Resistance

Prepared by Mark on October 5, 2023  
Based on a handout by Aaron Anderson

---

## Part 1: Random Walks

Consider the graph below. A particle sits on some node  $n$ . Every second, this particle moves left or right with equal probability. Once it reaches node  $A$  or  $B$ , it stops.

We would like to compute the probability of our particle stopping at node  $A$ .

In other words, we want a function  $P(n) : N \rightarrow [0, 1]$  that returns the probability that our particle stops at  $A$ , where  $N$  is the set of nodes in  $G$ .



### Problem 1:

What are  $P(A)$  and  $P(B)$  in the graph above?

Note that these values hold for all graphs.

### Problem 2:

Find an expression for  $P(x)$  in terms of  $P(y)$  and  $P(A)$ .

Find an expression for  $P(y)$  in terms of  $P(x)$  and  $P(B)$ .

### Problem 3:

Use the previous problems to find  $P(x)$  and  $P(y)$ .

**Problem 4:**

Say we have a graph  $G$  and a particle on node  $x$  with neighbors  $v_1, v_2, \dots, v_n$ .

Assume that our particle is equally likely to travel to each neighbor.

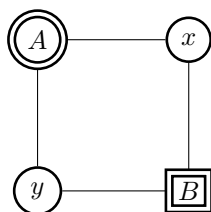
Find  $P(x)$  in terms of  $P(v_1), P(v_2), \dots, P(v_n)$ .

**Problem 5:**

How can we use Problem 4 to find  $P(n)$  for any  $n$ ?

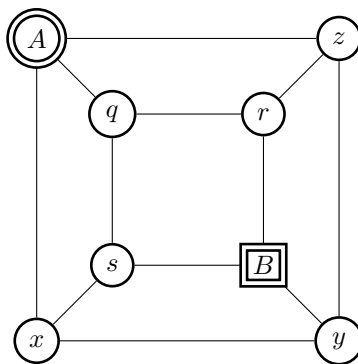
**Problem 6:**

Find  $P(n)$  for all nodes in the graph below.

**Problem 7:**

Find  $P(n)$  for all nodes in the graph below.

Note that this is the graph of a cube with  $A$  and  $B$  on opposing vertices.

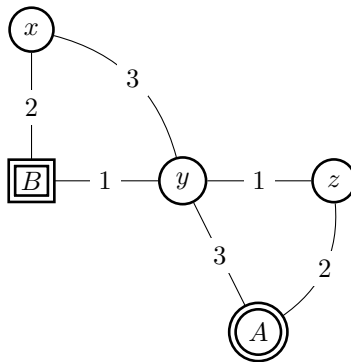


**Definition 8:**

Let us now take a look at weighted graphs. The problem remains the same: we want to compute the probability that our particle stops at node  $A$ , but our graphs will now feature weighted edges. The probability of our particle taking a certain edge is proportional to that edge's weight.

For example, if our particle is on node  $y$  of the graph below, it has a  $\frac{3}{8}$  probability of moving to  $x$  and a  $\frac{1}{8}$  probability of moving to  $z$ .

Note that  $3 + 3 + 1 + 1 = 8$ .

**Problem 9:**

Say a particle on node  $x$  has neighbors  $v_1, v_2, \dots, v_n$  with weights  $w_1, w_2, \dots, w_n$ . The edge  $(x, v_1)$  has weight  $w_1$ . Find  $P(x)$  in terms of  $P(v_1), P(v_2), \dots, P(v_n)$ .

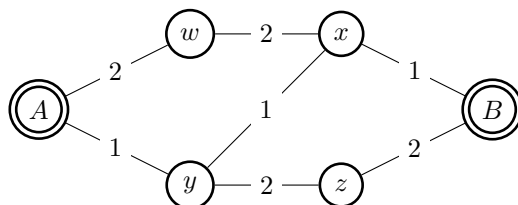
**Problem 10:**

Consider the following graph. Find  $P(x)$ ,  $P(y)$ , and  $P(z)$ .

**Problem 11:**

Consider the following graph.

What expressions can you find for  $P(w)$ ,  $P(x)$ ,  $P(y)$ , and  $P(z)$ ?



Solve this system of equations.

*Hint:* Use symmetry.  $P(w) = 1 - P(z)$  and  $P(x) = 1 - P(y)$ . Why?

## Part 2: Circuits

An *electrical circuit* is a graph with a few extra properties, called *current*, *voltage*, and *resistance*.

- **Voltage** is a function  $V(n) : N \rightarrow \mathbb{R}$  that assigns a number to each node of our graph.  
In any circuit, we pick a “ground” node, and define the voltage<sup>1</sup> there as 0.  
We also select a “source” node, and define its voltage as 1.  
Intuitively, you could say we’re connecting the ends of a 1-volt battery to our source and ground nodes.
- **Current** is a function  $I(e^\rightarrow) : N \times N \rightarrow \mathbb{R}$  that assigns a number to each *oriented edge*  $e^\rightarrow$  in our graph. An “oriented edge” is just an ordered pair of nodes  $(n_1, n_2)$ .  
Current through an edge  $(a, b)$  is a measure of the flow of charge from  $a$  to  $b$ .  
Naturally,  $I(a, b) = -I(b, a)$ .
- **Resistance** is a function  $R(e) : N \times N \rightarrow \mathbb{R}_0^+$  that represents a certain edge’s resistance to the flow of current through it.  
Resistance is a property of each *link* between nodes, so order doesn’t matter:  $R(a, b) = R(b, a)$ .

It is often convenient to compare electrical circuits to systems of pipes. Say we have a pipe from point  $A$  to point  $B$ . The size of this pipe represents resistance (smaller pipe  $\implies$  more resistance), the pressure between  $A$  and  $B$  is voltage, and the speed water flows through it is to current.

### Definition 12: Ohm’s law

With this “pipe” analogy in mind, you may expect that voltage, current, and resistance are related: if we make our pipe bigger (and change no other parameters), we’d expect to see more current. This is indeed the case! Any circuit obeys *Ohm’s law*, stated below:

$$V(a, b) = I(a, b) \times R(a, b)$$

$V(a, b)$  is the voltage between nodes  $a$  and  $b$ . If this doesn’t make sense, read the footnote below. In this handout, it will be convenient to write  $V(a, b)$  as  $V(a) - V(b)$ .

### Definition 13: Kirchoff’s law

The second axiom of electrical circuits is also fairly simple. *Kirchoff’s law* states that the sum of all currents connected to a given edge is zero. You can think of this as “conservation of mass”: nodes in our circuit do not create or destroy electrons, they simply pass them around to other nodes.

Formally, we can state this as follows:

Let  $x$  be a node in our circuit and  $B_x$  the set of its neighbors. We then have

$$\sum_{b \in B_x} I(x, b) = 0$$

which must hold at every node **except the source and ground vertices**.

*Hint:* Keep this exception in mind, it is used in a few problems later on.

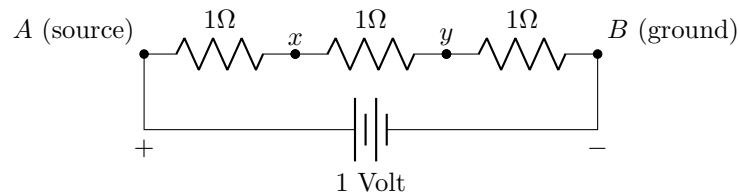
---

<sup>1</sup>In the real world, voltage is always measured *between two points* on a circuit. Voltage is defined as the *difference* in electrical charge between two points. Here, all voltages are measured with respect to our “ground” node. This detail isn’t directly relevant to the problems in this handout, so you mustn’t worry about it today. Just remember that the electrical definitions here are a significant oversimplification of reality.

Consider the circuit below. This is the graph from Problem 1, turned into a circuit by:

- Replacing all edges with  $1\Omega$  resistors
- Attaching a 1 volt battery between  $A$  and  $B$

Note that the battery between  $A$  and  $B$  isn't really an edge. It exists only to create a potential difference between the two nodes.



**Problem 14:**

From the circuit diagram above, we immediately know that  $V(A) = 1$  and  $V(B) = 0$ . What equations related to the currents out of  $x$  and  $y$  does Kirchhoff's law give us?

**Problem 15:**

Use Ohm's law to turn the equations from Problem 14 into equations about voltage and resistance. Find an expression for  $V(x)$  and  $V(y)$  in terms of other voltages, then solve the resulting system of equations. Does your result look familiar?

## Part 3: The Equivalence

In the last problem, we found that the equations for  $V(x)$  were the same as the equations for  $P(x)$  on the same graph. It turns out that this is true in general: problems about voltage in circuits directly correspond to problems about probability in graphs. We'll spend the next section proving this fact.

### Definition 16:

For the following problems, *conductance* will be more convenient than resistance.

The definition of conductance is quite simple:

$$C(a, b) = \frac{1}{R(a, b)}$$

*Aside:* Resistance is usually measured in Ohms, denoted  $\Omega$ .

A few good-natured physicists came up with the “mho” (denoted  $\mathfrak{U}$ ) as a unit of conductance, which is equivalent to an inverse Ohm. Unfortunately, NIST discourages the use of Mhos in favor of the equivalent (and less amusing) “Siemens.”

### Problem 17:

Let  $x$  be a node in a graph.

Let  $B_x$  be the set of  $x$ 's neighbors,  $w(x, y)$  the weight of the edge between nodes  $x$  and  $y$ , and  $W_x$  the sum of the weights of all edges connected to  $x$ .

We saw earlier that the probability function  $P$  satisfies the following sum:

$$P(x) = \sum_{b \in B_x} \left( P(b) \times \frac{w(x, b)}{W_x} \right)$$

This was never explicitly stated, but is noted in Definition 8.

Use Ohm's and Kirchoff's laws to show that the voltage function  $V$  satisfies a similar sum:

$$V(x) = \sum_{b \in B_x} \left( V(b) \times \frac{C(x, b)}{C_x} \right)$$

where  $C(x, b)$  is the conductance of edge  $(x, b)$  and  $C_x$  is the sum of the conductances of all edges connected to  $x$ .



Thus, if  $w(a, b) = C(a, b)$ ,  $P$  and  $V$  satisfy the same system of linear equations. To finish proving that  $P = V$ , we now need to show that there can only be one solution to this system. We will do this in the next two problems.

**Problem 18:**

Let  $q$  be a solution to the following equations, where  $x \neq a, b$ .

$$q(x) = \sum_{b \in B_x} \left( q(b) \times \frac{w(x, b)}{W_x} \right)$$

Show that the maximum and minimum of  $q$  are  $q(a)$  and  $q(b)$  (not necessarily in this order).

**Problem 19:**

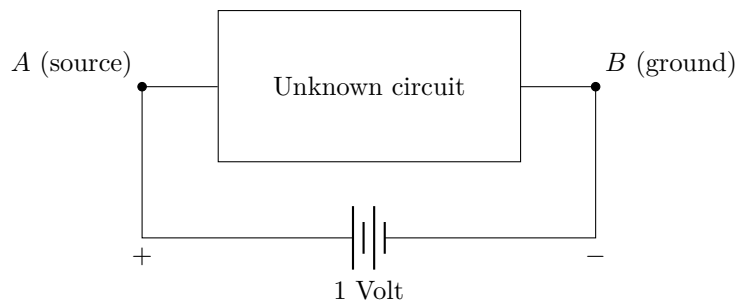
Let  $p$  and  $q$  be functions that solve our linear system and satisfy  $p(A) = q(A) = 1$  and  $p(B) = q(B) = 0$ .

Show that the function  $p - q$  satisfies the equations in Problem 18, and that  $p(x) - q(x) = 0$  for every  $x$ . Note that  $p(x) - q(x) = 0 \ \forall x \implies p = q$

## Part 4: Effective Resistance

As we have seen, calculating the properties of a circuit by creating an equation for each vertex is a fairly time-consuming ordeal. Fortunately, there is a better strategy we can use.

Consider a graph (or a circuit) with source and ground vertices. All parts of the circuit that aren't these two vertices are hidden inside a box, as shown below:



What do we know about this box? If this was a physical system, we'd expect that the current flowing out of  $A$  is equal to the current flowing into  $B$ .

### Problem 20:

Using Kirchoff's law, show that the following equality holds.

Remember that we assumed Kirchoff's law holds only at nodes other than  $A$  and  $B$ .

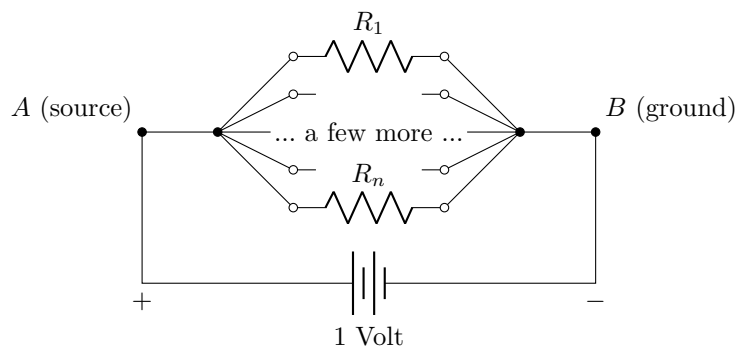
*Note:* As before,  $B_x$  is the set of neighbors of  $x$ . Naturally,  $B_B$  is the set of neighbors of  $B$ .

$$\sum_{b \in B_A} I(S, b) = \sum_{b \in B_B} I(b, B)$$

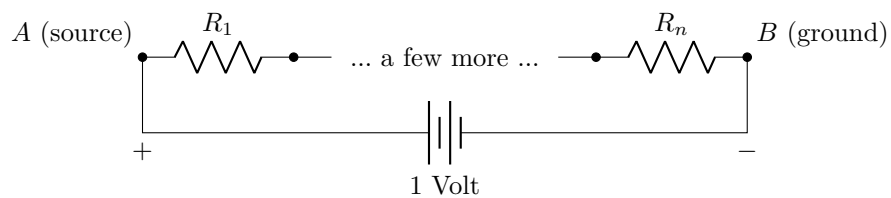
If we call this current  $I_A = \sum_{b \in B_A} I(A, b)$ , we can pretend that the box contains only one resistor, carrying  $I_A$  units of current. Using this information and Ohm's law, we can calculate the *effective resistance* of the box.

**Problem 21: Resistors in parallel**

Using Ohm's law and Kirchoff's law, calculate the effective resistance  $R_{\text{eff}}$  of the circuit below.

**Problem 22: Resistors in series**

Using Ohm's law and Kirchoff's law, calculate the effective resistance  $R_{\text{total}}$  of the circuit below.



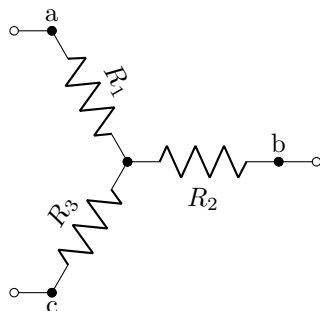
We can now use effective resistance to simplify complicated circuits. Whenever we see the above constructions (resistors in parallel or in series) in a graph, we can replace them with a single resistor of appropriate value.

**Problem 23:**

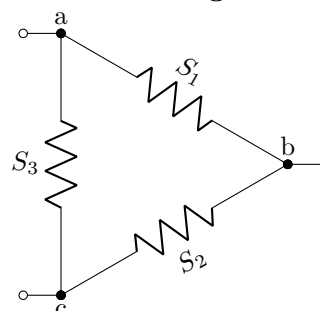
Consider the following circuits. Show that the triangle has the same effective resistance as the star if

- $x = R_1 R_2 + R_1 R_3 + R_2 R_3$
- $S_1 = x/R_3$
- $S_2 = x/R_1$
- $S_3 = x/R_2$

**The star:**



**The triangle:**



**Problem 24:**

Suppose we construct a circuit by connecting the  $2^n$  vertices of an  $n$ -dimensional cube with  $1\Omega$  resistors. If we place  $A$  and  $B$  at opposing vertices, what is the effective resistance of this circuit?

**Bonus:** As  $n \rightarrow \infty$ , what happens to  $R_{\text{eff}}$ ?

*Note:* Leave your answer as a sum.