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# Random Walks and Resistance

Prepared by Mark on April 3, 2024  
Based on a handout by Aaron Anderson

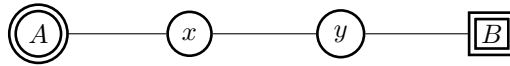
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## Part 1: Random Walks

Consider the graph below. A particle sits on some node  $n$ . Every second, this particle moves left or right with equal probability. Once it reaches node  $A$  or  $B$ , it stops.

We would like to compute the probability of our particle stopping at node  $A$ .

In other words, we want a function  $P : \text{Nodes} \rightarrow [0, 1]$  that maps each node of the graph to the probability that our particle stops at  $A$ .



### Problem 1:

What are  $P(A)$  and  $P(B)$  in the graph above?

Note that these values hold for all graphs.

### Problem 2:

Find an expression for  $P(x)$  in terms of  $P(y)$  and  $P(A)$ .

Find an expression for  $P(y)$  in terms of  $P(x)$  and  $P(B)$ .

### Problem 3:

Use the previous problems to find  $P(x)$  and  $P(y)$ .

**Problem 4:**

Say we have a graph  $G$  and a particle on node  $x$  with neighbors  $v_1, v_2, \dots, v_n$ .

Assume that our particle is equally likely to travel to each neighbor.

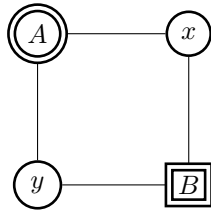
Find  $P(x)$  in terms of  $P(v_1), P(v_2), \dots, P(v_n)$ .

**Problem 5:**

In general, how do we find  $P(n)$  for any node  $n$ ?

**Problem 6:**

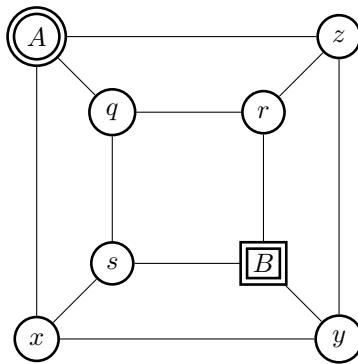
Find  $P(n)$  for all nodes in the graph below.



**Problem 7:**

Find  $P(n)$  for all nodes in the graph below.

Note that this is the graph of a cube with  $A$  and  $B$  on opposing vertices.

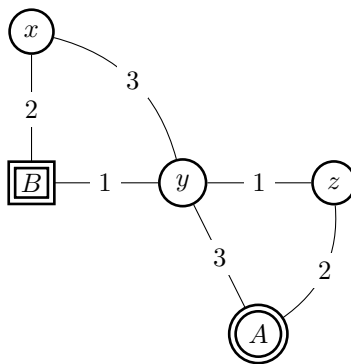


**Definition 8:**

Let us now take a look at weighted graphs. The problem remains the same: we want to compute the probability that our particle stops at node  $A$ , but our graphs will now feature weighted edges. The probability of our particle taking a certain edge is proportional to that edge's weight.

For example, if our particle is on node  $y$  of the graph below, it has a  $\frac{3}{8}$  probability of moving to  $x$  and a  $\frac{1}{8}$  probability of moving to  $z$ .

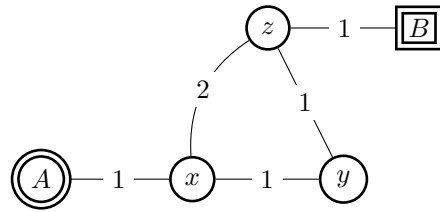
Note that  $3 + 3 + 1 + 1 = 8$ .

**Problem 9:**

Say a particle on node  $x$  has neighbors  $v_1, v_2, \dots, v_n$  with weights  $w_1, w_2, \dots, w_n$ . The edge  $(x, v_1)$  has weight  $w_1$ . Find  $P(x)$  in terms of  $P(v_1), P(v_2), \dots, P(v_n)$ .

**Problem 10:**

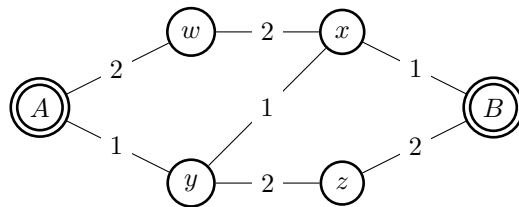
Consider the following graph. Find  $P(x)$ ,  $P(y)$ , and  $P(z)$ .



**Problem 11:**

Consider the following graph.

What expressions can you find for  $P(w)$ ,  $P(x)$ ,  $P(y)$ , and  $P(z)$ ?



Solve this system of equations.

*Hint:* Use symmetry.  $P(w) = 1 - P(z)$  and  $P(x) = 1 - P(y)$ . Why?

## Part 2: Circuits

An *electrical circuit* is a graph with a few extra properties, called *current*, *voltage*, and *resistance*. In the definitions below, let  $X$  be the set of nodes in a circuit.

- **Voltage** is a function  $V : X \rightarrow \mathbb{R}$  that assigns a number to each node of our graph. In any circuit, we pick a “ground” node, and define the voltage<sup>1</sup> there as 0. We also select a “source” node, and define its voltage as 1. Intuitively, you could say we’re connecting the ends of a 1-volt battery to our source and ground nodes.
- **Current** is a function  $I : X^2 \rightarrow \mathbb{R}$  that assigns a number to each *oriented edge* in our graph. An “oriented edge” is just an ordered pair of nodes  $(n_1, n_2)$ . Current through an edge  $(a, b)$  is a measure of the flow of charge from  $a$  to  $b$ . Naturally,  $I(a, b) = -I(b, a)$ .
- **Resistance** is a function  $R : X^2 \rightarrow \mathbb{R}_0^+$  that represents a certain edge’s resistance to the flow of current through it. Resistance is a property of each *link* between nodes, so order doesn’t matter:  $R(a, b) = R(b, a)$ .

It is often convenient to compare electrical circuits to systems of pipes. Say we have a pipe from point  $A$  to point  $B$ . The size of this pipe represents resistance (smaller pipe  $\implies$  more resistance), the pressure between  $A$  and  $B$  is voltage, and the speed water flows through it is to current.

### Definition 12: Ohm’s law

With this “pipe” analogy in mind, you may expect that voltage, current, and resistance are related: if we make our pipe bigger (and change no other parameters), we’d expect to see more current. This is indeed the case! Any circuit obeys *Ohm’s law*, stated below:

$$V(a, b) = I(a, b) \times R(a, b)$$

This handout uses two notations for voltage: two-variable  $V(a, b)$  and one-variable  $V(a)$ . The first represents the voltage between points  $a$  and  $b$ , better reflecting reality (see the footnote below). The second measures the voltage between  $a$  and ground, and is more convenient to use in equations. **Try to use the single-variable notation in your equations.** Convince yourself that  $V(a, b) = V(a) - V(b)$ .

### Definition 13: Kirchoff’s law

The second axiom of electrical circuits is also fairly simple. *Kirchoff’s law* states that the sum of all currents connected to a given edge is zero. You can think of this as “conservation of mass”: nodes in our circuit do not create or destroy electrons, they simply pass them around to other nodes.

Formally, we can state this as follows:

Let  $x$  be a node in our circuit and  $N_x$  the set of its neighbors. We then have

$$\sum_{b \in N_x} I(x, b) = 0$$

which must hold at every node **except the source and ground vertices.**

*Hint:* Keep this exception in mind, it is used in a few problems later on.

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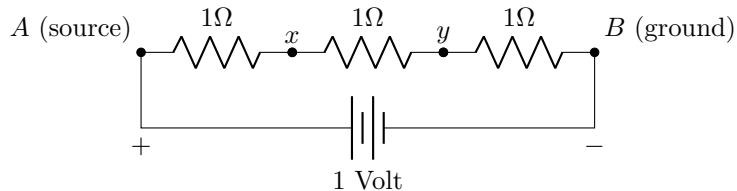
<sup>1</sup>In the real world, voltage is always measured *between two points* on a circuit. Voltage is defined as the *difference* in electrical charge between two points. Hence, voltage is a function of two nodes.

Note that this is different than current and resistance, which aren’t functions of two arbitrary nodes — rather, they are functions of *edges* (i.e., two adjacent nodes).

Consider the circuit below. **This the graph from Problem 1**, turned into a circuit by:

- Replacing all edges with  $1\Omega$  resistors
- Attaching a 1 volt battery between  $A$  and  $B$

Note that the battery between  $A$  and  $B$  isn't really an edge. It exists only to create a potential difference between the two nodes.



**Problem 14:**

From the circuit diagram above, we immediately know that  $V(A) = 1$  and  $V(B) = 0$ .

What equations related to the currents out of  $x$  and  $y$  does Kirchoff's law give us?

*Hint:* Current into  $x$  = current out of  $x$

**Problem 15:**

Use Ohm's law to turn the equations from Problem 14 into equations about voltage and resistance. Find an expression for  $V(x)$  and  $V(y)$  in terms of other voltages, then solve the resulting system of equations. Does your result look familiar?

### Part 3: The Equivalence

In the last problem, we found that the equations for  $V(x)$  were the same as the equations for  $P(x)$  on the same graph. It turns out that this is true in general: problems about voltage in circuits directly correspond to problems about probability in graphs. We'll spend the next section proving this fact.

**Definition 16:**

For the following problems, *conductance* will be more convenient than resistance. The definition of conductance is quite simple:

$$C(a, b) = \frac{1}{R(a, b)}$$

*Aside:* Resistance is usually measured in Ohms, denoted  $\Omega$ .

A few good-natured physicists came up with the “mho” (denoted  $\mathcal{U}$ ) as a unit of conductance, which is equivalent to an inverse Ohm. Unfortunately, NIST discourages the use of Mhos in favor of the equivalent (and less amusing) “Siemens.”

**Problem 17:**

Let  $x$  be a node in a graph.

Let  $N_x$  be the set of  $x$ 's neighbors,  $w(x, y)$  the weight of the edge between nodes  $x$  and  $y$ , and  $W_x$  the sum of the weights of all edges connected to  $x$ .

We saw earlier that the probability function  $P$  satisfies the following sum:

$$P(x) = \sum_{b \in N_x} \left( P(b) \times \frac{w(x, b)}{W_x} \right)$$

This was never explicitly stated, but is noted in Definition 8.

Use Ohm's and Kirchoff's laws to show that the voltage function  $V$  satisfies a similar sum:

$$V(x) = \sum_{b \in N_x} \left( V(b) \times \frac{C(x, b)}{C_x} \right)$$

where  $C(x, b)$  is the conductance of edge  $(x, b)$  and  $C_x$  is the sum of the conductances of all edges connected to  $x$ .



Thus, if  $w(a, b) = C(a, b)$ ,  $P$  and  $V$  satisfy the same system of linear equations. To finish proving that  $P = V$ , we now need to show that there can only be one solution to this system. We will do this in the next two problems.

**Problem 18:**

Let  $q$  be a solution to the following equations, where  $x \neq a, b$ .

$$q(x) = \sum_{b \in N_x} \left( q(b) \times \frac{w(x, b)}{W_x} \right)$$

Show that the maximum and minimum of  $q$  are  $q(a)$  and  $q(b)$  (not necessarily in this order).

**Problem 19:**

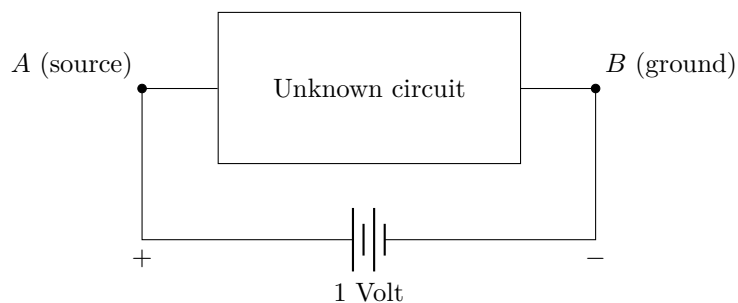
Let  $p$  and  $q$  be functions that solve our linear system and satisfy  $p(A) = q(A) = 1$  and  $p(B) = q(B) = 0$ .

Show that the function  $p - q$  satisfies the equations in Problem 18, and that  $p(x) - q(x) = 0$  for every  $x$ . Note that  $p(x) - q(x) = 0 \forall x \implies p = q$

## Part 4: Effective Resistance

As we have seen, calculating the properties of a circuit by creating an equation for each vertex is a fairly time-consuming ordeal. Fortunately, there is a better strategy we can use.

Consider a graph (or a circuit) with source and ground vertices. All parts of the circuit that aren't these two vertices are hidden inside a box, as shown below:



What do we know about this box? If this was a physical system, we'd expect that the current flowing out of  $A$  is equal to the current flowing into  $B$ .

### Problem 20:

Using Kirchoff's law, show that the following equality holds.

Remember that we assumed Kirchoff's law holds only at nodes other than  $A$  and  $B$ .

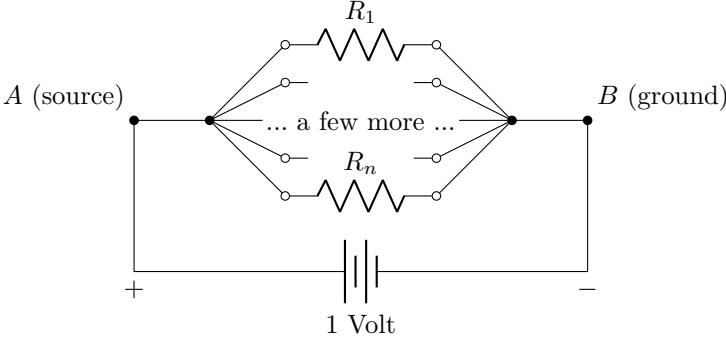
*Note:* As before,  $N_x$  is the set of neighbors of  $x$ .

$$\sum_{b \in N_A} I(A, b) = \sum_{b \in N_B} I(b, B)$$

If we call this current  $I_A = \sum_{b \in N_A} I(A, b)$ , we can pretend that the box contains only one resistor, carrying  $I_A$  units of current. Using this information and Ohm's law, we can calculate the *effective resistance* of the box.

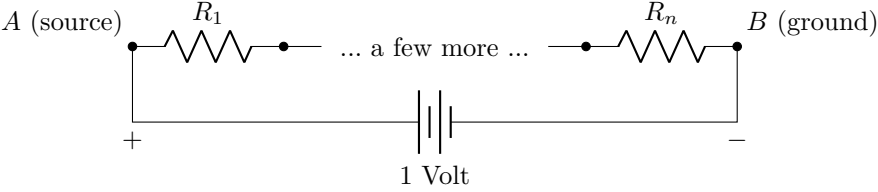
**Problem 21: Resistors in parallel**

Using Ohm's law and Kirchoff's law, calculate the effective resistance  $R_{\text{eff}}$  of the circuit below.



**Problem 22: Resistors in series**

Using Ohm's law and Kirchoff's law, calculate the effective resistance  $R_{\text{total}}$  of the circuit below.

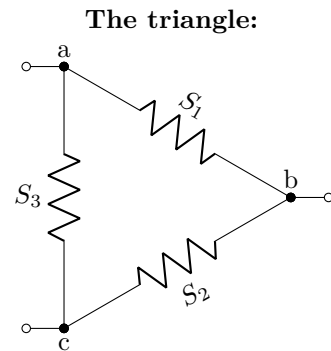
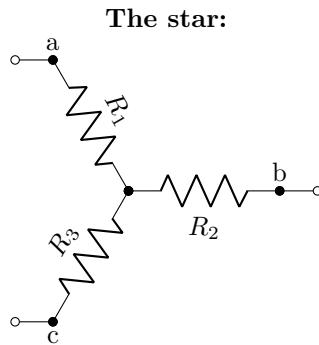


We can now use effective resistance to simplify complicated circuits. Whenever we see the above constructions (resistors in parallel or in series) in a graph, we can replace them with a single resistor of appropriate value.

**Problem 23:**

Consider the following circuits. Show that the triangle has the same effective resistance as the star if

- $x = R_1 R_2 + R_1 R_3 + R_2 R_3$
- $S_1 = x/R_3$
- $S_2 = x/R_1$
- $S_3 = x/R_2$



**Problem 24:**

Suppose we construct a circuit by connecting the  $2^n$  vertices of an  $n$ -dimensional cube with  $1\Omega$  resistors. If we place  $A$  and  $B$  at opposing vertices, what is the effective resistance of this circuit?

**Bonus:** As  $n \rightarrow \infty$ , what happens to  $R_{\text{eff}}$ ?

*Note:* Leave your answer as a sum.