Intermediate 1 **Oct 1, 2023**

Let's Count!

1 Probability Review

Recall that a *probability*, also known as a *chance*, is a number showing how likely some event is to happen. Let us call the event X. Then the probability of X taking place,

$$P(X) = \frac{\text{The number of the outcomes such that } X \text{ happens.}}{\text{The number of all the possible outcomes.}}$$

Problem 1 *A standard 6-sided die has exactly 6 faces, each has a distinct number from 1 to 6 labelled. Imagine you throw one die.*

- *What is the probability that you get the number 1?*
- *What is the probability that the number you get is odd?*
- *What is the probability that the number is at least 3?*

Problem 2 *In the dice game we played last week, you threw two dice at the same time and summed the numbers up.*

- *List all possible outcomes of the two dice such that the sum of two dice is 2. What is the probability that you get a sum of 2?*

- *List all possible outcomes of the two dice such that the sum of two dice is 3. What is the probability that you get a sum of 3?*

Problem 3 *Imagine you now throw five dice at the same time.*

- *What is the probability that you get 5 as the sum of all dice? (Hint:* $6^5 = 7776$*.)*

- *What is the probability that you get 6 as the sum?*

2 Intro to Combinatorics

Problem 4 *How many ways are there to put three marbles of different colors in a row?*

Definition 1 *The product of all the natural numbers from 1 through n is called n factorial.*

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

For example, $3! = 6$. It is a useful convention to set $0! = 1$.

Problem 5 *Compute the following numbers.*

$a.\quad 5! =$ $\qquad\qquad$ $b.\quad 6! =$

Problem 6 *How many ways are there to put $n + 1$ marbles of different colors in a row?*

Problem 7 *Compute the following numbers.*

$a.\quad \dfrac{5!}{4!} =$ $\qquad\qquad$ $b.\quad \dfrac{100!}{98!} =$

Problem 8 *There are 10 marbles of different colors in a box. How many ways are there to put 6 of them in a row? Write down the answer using the $n!$ notation.*

Problem 9 *Let k and n be natural numbers such that $k \leq n$. There are n marbles of different colors in a box. How many ways are there to put k of them in a row?*

Definition 2 *A way to choose k objects out of n so that the order of the chosen objects matters is called a permutation.*

As proven in Problem 9, the number of permutations is given by the following formula.

$$P(n, k) = \frac{n!}{(n-k)!} \tag{1}$$

Definition 3 *A way to choose k objects out of n so that the order of the chosen objects does not matter is called a combination.*

Example 1 *A poker hand is any combination of 5 cards out of the deck of 52. Let us compute the number of the possible poker hands. First, there are*

$$P(52,5) = \frac{52!}{(52-5)!} = 48 \times 49 \times 50 \times 51 \times 52 = 311,875,200$$

ways to put 5 cards out of 52 in a row. However, the order of the cards in a hand does not matter. There are 5! *ways to order 5 cards. (Why?) Dividing the above number by* 5! *gives us the number of the possible poker hands.*

$$C(52,5) = \frac{P(52,5)}{5!} = \frac{52!}{(52-5)!\,5!} = 2,598,960$$

A more modern, but less convenient, notation for the number of combinations is given on the left-hand side of the following formula.

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = C(n,k) \tag{2}$$

The above reads as n *choose* k. For the reason that will become clear in Problem 19, the formula is also known as a *binomial coefficient.*

Problem 10 *Compute the following numbers.*

$$a. \quad \binom{5}{0} =$$

b. $\binom{5}{1} =$

c. $\binom{10}{3} =$

d. $\binom{10}{7} =$

Problem 11 *Prove the following property of the binomial coefficients.*

$$\binom{n}{k} = \binom{n}{n-k} \tag{3}$$

Problem 12 *Prove the following formulae.*

a. $\binom{4}{2} + \binom{4}{3} = \binom{5}{3}$

b. $\binom{5}{3} + \binom{5}{4} = \binom{6}{4}$

Problem 13 *Prove that the following formula holds for any $n \in \mathbb{N}$ and for any $k = 0, 1, 2, \ldots, n$.*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{4}$$

Formula 4 explains why it is possible to arrange the binomial coefficients in the following table known as *Pascal's triangle*.

$$
\begin{array}{lccccccccccc}
n = 0: & & & & & & 1 & & & & & \\
n = 1: & & & & & 1 & & 1 & & & & \\
n = 2: & & & & 1 & & 2 & & 1 & & & \\
n = 3: & & & 1 & & 3 & & 3 & & 1 & & \\
n = 4: & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n = 5: & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
n = 6: & & & & & & & & & & & \\
n = 7: & & & & & & & & & & &
\end{array}
$$

Problem 14 *In the table above, fill in the entries of the Pascal's triangle for $n = 6$ and 7.*

The upper-case Greek letter Σ (Sigma) is used in math as a notation for a sum. For example,

$$\sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

Problem 15 *Expand the following formula.*

$$\sum_{k=0}^{3} \binom{3}{k} x^{3-k} y^k =$$

Problem 16 *Expand and simplify the following.*

$(x + y)^3 =$

Compare your result to that in Problem 15.

Problem 17 *Expand the following formula.*

$$\sum_{k=0}^{4} \binom{4}{k} x^{4-k} y^k =$$

Problem 18 *Use the answer to Problem 16 to expand and simplify the following.*

$(x + y)^4 =$

Compare your result to that in Problem 17.

Problem 19 *Prove the following statement known as the binomial formula.*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\, y^k = \\ x^n + nx^{n-1}y + \tbinom{n}{2}x^{n-2}y^2 + \tbinom{n}{3}x^{n-3}y^3 + \ldots + y^n \tag{5}$$

Note that the corresponding line of the Pascal's triangle gives you all the binomial coefficients for the binomial of power n.

Problem 20 *Use the binomial formula to expand the following polynomial.*

$$(x+1)^5 =$$

Problem 21 *Use the Pascal triangle to find the sum of all the binomial coefficients for the following n.*

$n = 0$ **:**

$n = 1$ **:**

$n = 2$ **:**

$n = 3$ **:**

$n = 4$ **:**

$n = 5$ **:**

Can you guess the general pattern?

Problem 22 *Prove the following formula.*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{6}$$

3 Back to Probability!

Problem 23 *We toss a coin 4 times. How many ways are there to get the following outcomes?*

0 heads:

1 head:

2 heads:

3 heads:

4 heads:

Compare the above numbers to the Pascal's triangle line for $n = 4$. *Can you explain what you see?*

Suppose that the coin you toss is not necessarily fair. At every toss, the chance to get a head is p.

Problem 24 *What is the chance to get a tail?*

Problem 25 *You toss the above coin 7 times. What is the chance that you get 3 heads?*

Problem 26 *You toss the above coin 7 times. What is the chance that you get no more than 3 heads?*

Problem 27 *Assume that the coin in Problem 26 was a fair one. What is the probability that you get no more than 3 heads?*

Problem 28 *You toss a fair coin 12 times. What is the chance that you get 5 tails?*

Problem 29 *To win a jackpot in the Mega Millions lottery, you have to guess right 5 numbers from a pool of numbers from 1 to 56 and an additional Mega Ball number from a second pool of numbers from 1 to 46. What is a chance to hit the jackpot, if you purchase one lottery ticket?*

Problem 30 *The price of one lottery ticket for the Mega Millions lottery is \$1. How much money do you need to invest to have a* 0.5 *chance of winning the jackpot?*

Problem 31 *A club has 10 members, 5 boys and 5 girls. Two of the members are chosen at random. What is the probability that they are both girls?*

Problem 32 *Another club has 20 members, 12 boys and 8 girls. Two of the members are chosen at random. What is the probability that a boy and a girl are chosen?*

Problem 33 *What is the probability that a random arrangement of the letters in the word 'SEVEN' will have both E's next to each other?*

Problem 34 *2 vertices of an octagon are chosen at random. What is the probability that they are adjacent?*

Problem 35 *3 cards are chosen at random from a standard 52-card deck. What is the probability that they form a pair? (A 3-card hand is a "pair" if two of the cards match in rank but the third card is different. For example, 668 is a pair, but 999 is not.)*

Problem 36 *Perhaps the most common error in ranking poker hands involves juxtaposing the order of three of a kind and two pairs. Compare the probability of getting three of a kind (e.g. 22245) with that of getting two pairs (e.g. 22445). Can you offer an intuitive argument that would have allowed you to determine which is more likely without performing any calculation at all?*

Problem 37 *2 diagonals of a regular heptagon (a 7-sided polygon) are chosen. What is the probability that they intersect inside the heptagon?*

Problem 38 *I have 120 blocks. Each block is one of 2 different materials, 3 different colors, 4 different sizes, and 5 different shapes. No two blocks have exactly the same of all four properties. I take two blocks at random. What is the probability the two blocks have exactly two of these four properties the same?*