Olympiads Week 2

ORMC

10/8/23

This week, remember to *write down your solutions, as proofs.* You don't have to start by writing out a full proof to every problem you try, but once you've solved a problem or two, take a few minutes to write out a proof as if this was being graded at an Olympiad.

1 Practice Problems

Problem 1.1. A group of n people play a round-robin tournament - every team plays every other team exactly once, and each game ends in either a win or a loss. Show that it is possible to label the players $P_1, P_2, \ldots, P_n$ in such a way that P_1 defeated P_2, P_2 defeated P_3, and so on through P_{n-1} defeated P_n.

Problem 1.2. Show that if a round-robin tournament has an odd number of teams, it is possible for every team to win exactly half its games.

1.1 From Problem-Solving Through Problems

Problem 1.3. Find positive numbers n and $a_1, a_2, \ldots, a_n$ such that $a_1 + \cdots + a_n = 1000$ and the product $a_1 a_2 \ldots a_n$ is as large as possible.

Problem 1.4. Determine the number of odd binomial coefficients in the expansion of $(x+y)^{100}$.

1.2 From USSR Olympiad Problem Book

Problem 1.5. Prove that the product of four consecutive positive integers is one less than a perfect square.

Problem 1.6. Calculate the following sums:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n}$$

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{(n-2)(n-1)n}$$

$$\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{(n-3)(n-2)(n-1)n}$$

2 Competition Problems

Problem 2.1 (BAMO 2008 Problem 5)**.** N teams participated in a national basketball championship in which every two teams played exactly one game. Of the N teams, 251 are from California. It turned out that a Californian team Alcatraz is the unique Californian champion (Alcatraz has won more games against Californian teams than any other team from California). How- ever, Alcatraz ended up being the unique loser of the tournament because it lost more games than any other team in the nation! What is the smallest possible value for N?

Problem 2.2 (BAMO 2013 Problem 7)**.** Let $F_1, F_2, F_3 \ldots$ be the Fibonacci sequence, the sequence of positive integers with $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. A Fibonacci number is by definition a number appearing in this sequence. Let $P_1, P_2, P_3, \ldots$ be the sequence consisting of all the integers that are products of two Fibonacci numbers (not necessarily distinct), in increasing order. The first few terms are

$$1, 2, 3, 4, 5, 6, 8, 9, 10, 13, \ldots$$

since, for example $3 = 1 \cdot 3, 4 = 2 \cdot 2$, and $10 = 2 \cdot 5$. Consider the sequence D_n of successive differences of the P_n sequence, where $D_n = P_{n+1} - P_n$ for $n \geq 1$. The first few terms of D_n are

$$1, 1, 1, 1, 1, 2, 1, 1, 3, \ldots$$

Prove that every number in D_n is a Fibonacci number.

Problem 2.3 (USAMO 2008 Problem 1)**.** Prove that for each positive integer n, there are pairwise relatively prime integers $k_0, k_1 \ldots, k_n$, all strictly greater than 1, such that $k_0 k_1 \cdots k_n - 1$ is the product of two consecutive integers.

Problem 2.4 (USAMO 2003 Problem 6)**.** At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Problem 2.5 (USAMO 2011 Problem 6)**.** Let A be a set with $|A| = 225$, meaning that A has 225 elements. Suppose further that there are eleven subsets $A_1, \ldots, A_{11}$ of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \cdots \cup A_{11}| \geq 165$, and give an example for which equality holds.

Problem 2.6 (Putnam 2021 Problem A3)**.** Determine all positive integers N for which the sphere

$$x^2 + y^2 + z^2 = N$$

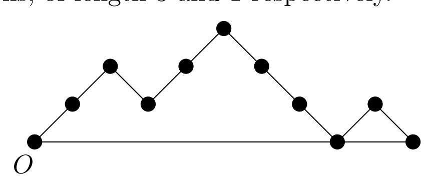
has an inscribed regular tetrahedron whose vertices have integer coordinates.

Problem 2.7 (Putnam 2003 Problem A1)**.** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

Problem 2.8 (Putnam 2003 Problem A5)**.** A Dyck n-path is a lattice path of n upsteps $(1, 1)$ and n downsteps $(1, -1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n - 1)$-paths.