Los Angeles Math Circle

Sets and Venn Diagrams

Vocabulary and notations

A *set* is a clearly defined collection of distinct objects. Note that this is not really a definition. To define means to explain in simpler terms. Instead, all we do is replacing one word, a *set*, by another, a *collection*. Plus, the meaning of the words *to clearly define* is not clearly defined. The problem is that the notion of a set is as fundamental as it is deep. It is impossible to explain it in simpler terms. The best we can do at the moment is to show a bunch of examples.

The set F of all the factors of the number 6 consists of four elements, six, three, two, and one.

$$F = \{6, 3, 2, 1\}$$

Note that we use curvy braces, $\{*\}$, to write down the set elements. In a set, the order of the elements does not matter,

$$F = \{1, 6, 2, 3\}.$$

Let us list the elements of the above set in the increasing order.

$$(1, 2, 3, 6)$$

As you can see, we have switched from the braces to round brackets, $(*)$, a.k.a. parentheses. The round brackets are used to write down elements of an *ordered set*, or a *list*. In a list, the order of the elements does matter. For example, let us consider two sets and two lists of letters, $S_1 = \{C, A, T\}$, $S_2 = \{A, C, T\}$ and $L_1 = (C, A, T)$, $L_2 = (A, C, T)$. Then $S_1 = S_2$, but $L_1 \neq L_2$.

The fact that the letter A is an element of the set S_1 is denoted as $A \in S_1$.

Looking at the sets $S_1 = \{C, A, T\}$ and $S_3 = \{A, C\}$, one can see that every element of S_3 is also an element of S_1. In this case, S_3 is called a *subset* of S_1. In mathematical language, we write this as $S_3 \subseteq S_1$. Any set is its own subset, $S_1 \subseteq S_1$, $S_3 \subseteq S_3$, etc.

Problem 1 *What is the set of colors of the United States flag?*

The set $\mathbb{N} = \{1, 2, 3, \ldots\}$ is called the set of *natural numbers*. It is a recent tendency to include zero in the set. This is very wrong for historical reasons. The highly non-trivial idea to use a special symbol for *nothing* occurred to humanity tens, if not hundreds, of thousands of years after people had learned to use natural numbers for counting. Zero is too complicated to be natural! In mathematical language, we write this as $0 \notin \mathbb{N}$. The latter reads formally as *zero does not belong to the set of natural numbers.*

The set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is called the set of *integral numbers*, or just *integers.* $\mathbb{N}$ is a subset of $\mathbb{Z}$. Recall that in the math language, we write this as $\mathbb{N} \subseteq \mathbb{Z}$. Zero is an integer, $0 \in \mathbb{Z}$.

A set without any elements is called the *empty set* and is denoted as $\emptyset$. The empty set is a subset of any set, including itself, $\emptyset \subseteq \emptyset$. Note that it is **the** empty set, not **an** empty set. There exists only one such.

Example 1 $\{$*A set of pigs that can fly by themselves.*$\} = \emptyset$

Problem 2 *Give your own description of the empty set.*

A subset of a set is called *proper* if it is not empty and is not equal to the original set. For example the set S_3 we have considered above is a proper subset of the set S_1. In mathematical language, we write this as $S_3 \subset S_1$. Likewise, $\mathbb{N} \subset \mathbb{Z}$.

Note 1 *The notations $\subset$ and $\subseteq$ for sets are analogous to $<$ and $\leq$ for numbers.*

Problem 3 *Let $S_1 = \{C, A, T\}$ and $S_2 = \{A, C, T\}$. Is S_1 a subset of S_2? Is S_1 a proper subset of S_2?*

Problem 4 *Write down all the proper subsets of the set of colors of the US flag. Do they form a set? A list?*

Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. The set $\mathbb{Q}$ of all the fractions $\frac{m}{n}$ in the reduced form, a.k.a. in lowest terms, is called the set of *rational numbers*. Recall that the fraction $\frac{m}{n}$ is in the reduced form if $|m|$ and n have no common factors except for 1. For example, $\frac{2}{3} \in \mathbb{Q}$. An integer m can be considered as a fraction $\frac{m}{1}$. Therefore, $\mathbb{Z} \subset \mathbb{Q}$.

Problem 5 *Is the number* 0.55 *rational? Why or why not?*

Problem 6 *Decide which of the following two fractions is greater without cross-multiplying or bringing to the common denominator.*

$$\frac{2017}{2018} \quad or \quad \frac{2018}{2019}$$

The number of elements in a set is called its *cardinality*. In math language, the cardinality of a set A is denoted as either $|A|$ or $card(A)$. We will use the first notation in the handout. For example, $|S_1| = 3$ for the set $S_1 = \{C, A, T\}$ considered earlier.

Problem 7 *Let U be the set of states in the United States. What is $|U|$?*

Problem 8 *What is $|\{0\}|$? Why?*

What is $|\emptyset|$?

Problem 9 *How many subsets, including the empty set and the set proper, has a set A of cardinality a. 2, b. 3, c. 4, d. 5? Do you see a pattern?*

Problem 10 *Given $|A| = n$, prove that the cardinality of the set of all the subsets of A is 2^n.*

The set of the elements that belong to the sets A and B is called the *intersection* of A and B and is denoted as $A \cap B$. Let us rewrite this definition completely in the math language.

$$A \cap B \stackrel{\text{def}}{=} \{x : x \in A \text{ and } x \in B\} \tag{1}$$

In this mathematical sentence, the colon reads as *such that.* Translating back into English, *the intersection of the sets A and B is defined as the set of the elements x such that x belongs to A and x belongs to B.*

Problem 11 *Let A be the set of all the even numbers, a.k.a. the integers divisible by 2. Let B be the set of all the integers divisible by 3. What is $A \cap B$?*

Problem 12 *What is $A \cap \emptyset$ for any set A?*

The following is the definition of the *union* of two sets, written down in the math language.

$$A \cup B \stackrel{\text{def}}{=} \{x : x \in A \text{ or } x \in B\} \tag{2}$$

Problem 13 *Translate (2) into English.*

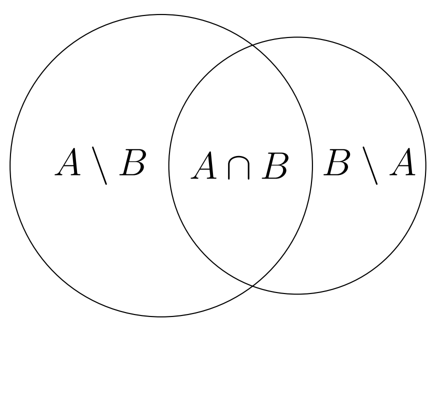
Problem 14 *What is $A \cup \emptyset$ for any set A?*

Problem 15 $S_1 = \{C, A, T\}$ *and* $S_2 = \{A, C, T\}$. *What is* $S_1 \cup S_2$*?*

Problem 16 *Give an example of two sets and of their union different from the ones used above.*

Venn diagrams and the inclusion-exclusion principle

The *difference of the sets* A and B, the set $A \setminus B$, is the set of all the elements of the set A that do not belong to the set B. The following picture, called a *Venn diagram*, helps to visualize the definition.



Problem 17 *Show the set* $A \cup B$ *on the Venn diagram above.*

Problem 18 *Let* A *be the set of spectators at a basketball game. Let* B *be the set of all the people at the game, spectators, coaches, stuff, etc., wearing caps. Describe in your own words the set* $A \setminus B$*.*

Problem 19 *Use the symbol* $\not\in$ *to write the definition of the set* $A \setminus B$ *in the math language.*

Problem 20 *How many integers in the set*

$$S = \{1, 2, 3, \ldots, 98, 99, 100\}$$

are not divisible by 3?

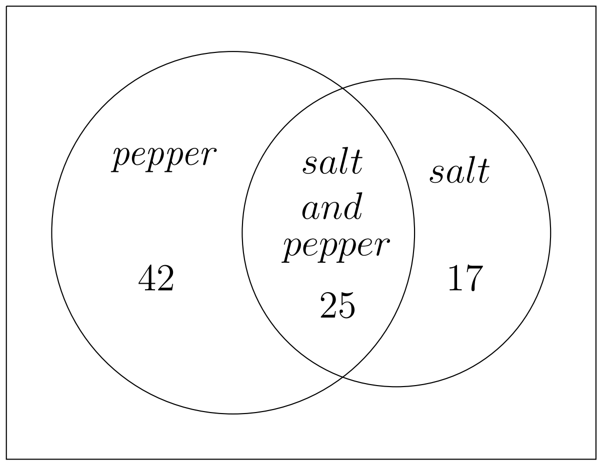
Two sets are called *disjoint*, if they have no elements in common. In other words, if the sets A and B are disjoint if and only if $A \cap B = \emptyset$.

Problem 21 *Give an example of two disjoint sets.*

Problem 22 *The sets A and B are disjoint. Draw the corresponding Venn diagram.*

Problem 23 $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$, $A \cap C = \emptyset$. *Draw the corresponding Venn diagram.*

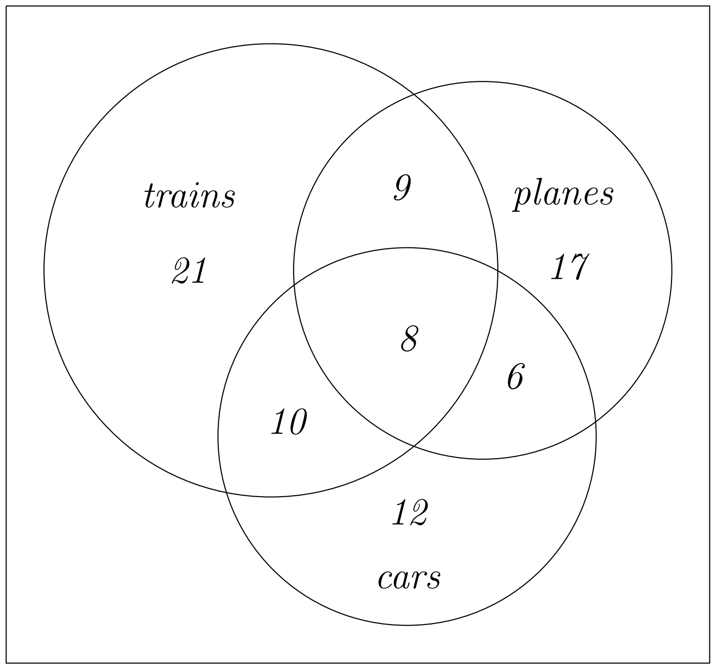
Problem 24 *Pablo asked 100 steak lovers whether they liked to put salt and pepper on their filet mignons.*



According to the Venn diagram above, how many put

- *salt?*
- *salt only?*
- *pepper only?*
- *salt and pepper?*
- *pepper?*
- *neither?*

Problem 25 *Gregory asked 100 kids whether they were collecting die-cast models of cars, trains, and airplanes.*



According to the Venn diagram above, how many kids were collecting

- *trains?*
- *planes?*
- *trains and planes?*

The problem continues to the next page.

- *trains and planes, but not cars?*

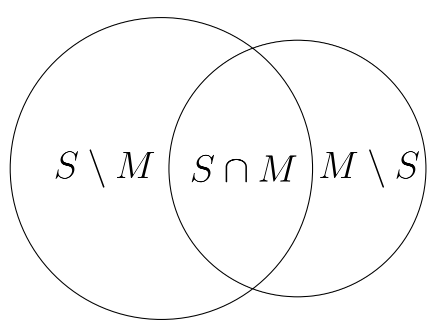
- *trains and cars, but not planes?*

- *all of them?*

- *neither of them?*

Example 2 *There are 120 sixth grade students in a US school, each of them taking at least one world language class, in addition to English. 86 students study Spanish, 38 learn Mandarin. How many students take both classes?*

Let S be the set of the students taking the Spanish class. Let M be the set of the students studying Mandarin. The following Venn diagram helps to visualize the problem.



We need to find the cardinality of the intersection of the two sets, $|S \cap M|$. Let us call it x.

$$|S \cap M| = x$$

Then $|S \setminus M| = 86 - x$ *and* $|M \setminus S| = 38 - x$. *Therefore,* $|S \setminus M| + |S \cap M| + |M \setminus S| = 86 - x + x + 38 - x = 120$. *The equation simplifies to* $124 - x = 120$. *Thus,* $x = 4$.

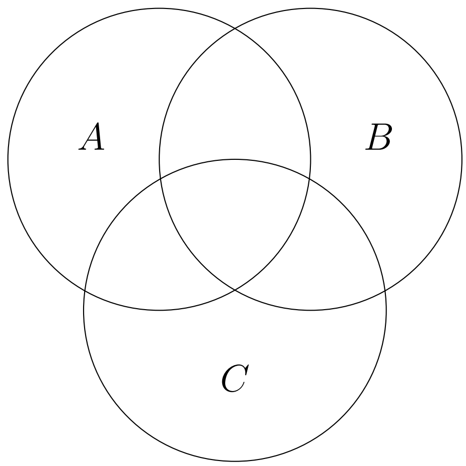
The above problem has a simpler solution. Adding up the numbers of the students taking Spanish and the students taking Mandarin, we count those who take both languages twice. Therefore, the sum $|S| + |M| = 86 + 38 = 124$ exceed the total number of the 6th grade students, 120, by $|S \cap M|$. Hence, $|S \cap M| = 4$. We just have seen the simplest case of a rather handy combinatorial technique, called the *inclusion-exclusion principle.*

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{3}$$

Problem 26 *Prove (3).*

The following useful formulas are proven by looking at the three-set Venn diagram below.

$$
\begin{aligned}
(A \cup B) \cap C &= (A \cap C) \cup (B \cap C) \\
(A \cap B) \cup C &= (A \cup C) \cap (B \cup C)
\end{aligned}
\tag{4}
$$



Problem 27 *Prove formulas (4).*

The following is the inclusion-exclusion principle for three sets, A, B, and C.

$$\begin{aligned} &|A \cup B \cup C| = |A| + |B| + |C| - \\ &-|A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \end{aligned} \tag{5}$$

Problem 28 *Among 20 students in a room, 9 study Mathematics, 10 study Science, and 8 take Music classes. 4 students study Mathematics and Science, 4 take Mathematics and Music, and 3 take Science and Music. 2 students take all the three subjects. How many students take at least one of the three courses? How many take none? Solve the problem in two (slightly) different ways, using a Venn diagram and using the inclusion-exclusion principle.*

Example 3 *How many integers in the set*

$$S = \{1, 2, 3, \ldots, 98, 99, 100\}$$

are not divisible by 2, 3, or 5?

To solve this problem, let us use the inclusion-exclusion principle for three sets to count the integers from S *that are divisible by either 2, 3, or 5. If the number of these integers is* n*, then the numbers of the integers from* S *not divisible by either 2, 3, or 5 is* $100 - n$.

Let A *be the set of the integers from* S *divisible by 2. Then* $|A| = 50$.

Let B *be the set of the integers from* S *divisible by 3. Solving Problem 20, we have found that* $|B| = 33$.

Let C *be the set of the integers from* S *divisible by 5. Then* $|C| = 20$.

The set $A \cap B$ *is the set of all the integers from* S *divisible by 6.* $|A \cap B| = 16$.

The set $A \cap C$ *is the set of all the integers from* S *divisible by 10.* $|A \cap C| = 10$.

The set $B \cap C$ *is the set of all the integers from* S *divisible by 15.* $|B \cap C| = 6$.

The set $A \cap B \cap C$ *is the set of all the integers from* S *divisible* *by 30.* $|A \cap B \cap C| = 3$.

The set $A \cup B \cup C$ *is the set of all the integers from* S *divisible by either 2, or 3, or 5. According to (5),* $|A \cup B \cup C| = 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$. *This way,* $n = 74$.

Finally, the number of the elements of S *not divisible by either 2, 3, or 5 equals* $100 - n = 100 - 74 = 26$.

For a relatively small number n, such as 100, you can simply go through all the integers 1 through n, determine whether each of them is or is not divisible by either 2, 3, or 5, and count the desirables. It is a bit more time-consuming for a greater number, like $n = 1,000,000$.

Homework Problem 1 *How many integers in the set*

$$S = \{1, 2, 3, \ldots, 999998, 999999, 1000000\}$$

are not divisible by 2, 3, or 5? Feel free to use a calculator for your computations.

Homework Problem 2 *Use formulas (4) to prove formula (5).*

Homework Problem 3 *Design and solve your own problem utilizing a three-set inclusion-exclusion principle.*

Applications of Venn diagrams to number theory

Problem 29 *Let* $A = \{\textit{divisors of } 10\}$*,* $B = \{\textit{divisors of } 18\}$*, and* $C = \{\textit{divisors of } 45\}$*.*

- *List the elements of* A*,* B*, and* C*.*

- *List the elements of* $A \cap B$*,* $A \cap C$*, and* $B \cap C$*.*

- *List the elements of* $A \cap B \cap C$*.*

- *Draw the Venn diagram for the sets* A*,* B*, and* C*.*

- *Find GCD(10,18,45), the greatest common divisor of 10, 18, and 45.*

Problem 30 *Let* $P = \{$*multiples of 6, from 6 to 75*$\}$*,*
$Q = \{$*multiples of 8, from 6 to 75*$\}$*, and*
$R = \{$*multiples of 9, from 9 to 75*$\}$*.*

- *List the elements of* P*,* Q*, and* R*.*

- *List the elements of* $P \cap Q$*,* $P \cap R$*, and* $Q \cap R$*.*

- *List the elements of* $P \cap Q \cap R$*.*

- *Draw the Venn diagram for the sets* P*,* Q*, and* R*.*

The problem continues to the next page.

- *Find LCM(6,8), the least common factor of 6 and 8.*

- *Find LCM(6,9).*

- *Find LCM(8,9).*

- *Find LCM(6,8,9).*

Homework Problem 4 *Use a Venn diagram to find GCD(12,21,28).*

Homework Problem 5 *Formulate the inclusion-exclusion principle for four sets, A, B, C, and D.*

Homework Problem 6 *How many integers in the set*

$$S = \{1, 2, 3, \ldots, 999998, 999999, 1000000\}$$

are not divisible by 2, 3, 5, or 7?

The following problem is unrelated to Venn diagrams, but it is related to number theory and is super-cool!

Homework Problem 7 *(Oldaque P. de Freitas Puzzle)*

Two ladies are sitting in a street café, talking about their children. One lady says that she has three daughters. The product of the girls' ages equals 36 and the sum of their ages is the same as the number of the house across the street. The second lady replies that this information is not enough to figure out the age of each child. The first lady agrees and adds that her oldest daughter has beautiful blue eyes. Then the second lady solves the puzzle. Please do the same.

Sets in geometry

An *ellipse* is a set of all the points in the plane such that the sum of their distances to a pair of points, called the ellipse's *foci*,[1] is a constant.

Problem 31 *Given a goat, a grass-covered flat meadow, two stakes, an ample amount of rope, and a collar, how would you make the goat graze out an ellipse in the grass?*

Problem 32 *What is an ellipse with two coinciding foci?*

Problem 33 *Use the notion of a set to define an important geometric object different from an ellipse.*

[1] plural of *focus*

Statement 1 *An angular bisector is the set of all the points of the angle equidistant from the angle's sides.*

Problem 34 *Prove Statement 1*

Problem 35 *Use the notion of a set to formulate and prove a Geometry statement.*

Self-test questions

The following questions will help you review the main concepts of the handout.

- What is a set?

- What is the difference between a set and a list?

- Does zero belong to the set of natural numbers? Why or why not?

- What is a proper subset of a set?

- Is the set of natural numbers a proper subset of the set of integers?

- What is a rational number?

- What is the cardinality of a set?

- What is an intersection of two sets?

- What is a union of two sets?

- What is the difference of two sets?

- What is the inclusion-exclusion principle for two sets?

- What is the inclusion-exclusion principle for three sets?

- What is the greatest common divisor of two natural numbers?
- What is the least common multiple of two natural numbers?
- How do Venn diagrams help to compute a GCD and LCM?
- How does the notion of a set help to define geometric objects?